Exact results and new insights for Models defined over Small-World Networks
First- and Second-order Phase Transitions

M. Ostilli[1,2] in collaboration with J. F. F. Mendes[1]

[1] Departamento de Física da Universidade de Aveiro, Portugal

[2] Center for Statistical Mechanics and Complexity, INFM-CNR SMC, Roma, Italy.

NetSci 2008 - Norwich

Outline

1. Introduction
   - Small-World Networks
   - Models on Small-World Networks
   - Aim: Investigate the Collective Behavior
2. The “Traditional” Methods
3. Our Method
   - An Effective Field Theory
   - Two Clear Examples
   - The Dual Phase Transition Scenario

Poissonian Small-World Networks

- Given an arbitrary graph $(\mathcal{L}_0, \Gamma_0)$ with N sites,
- We add $L = \frac{c \times N}{2}$ bonds uniformly over $\mathcal{L}_0$
- $\Rightarrow$ added connectivity Poisson distributed with mean c.

Poissonian Small-World Networks

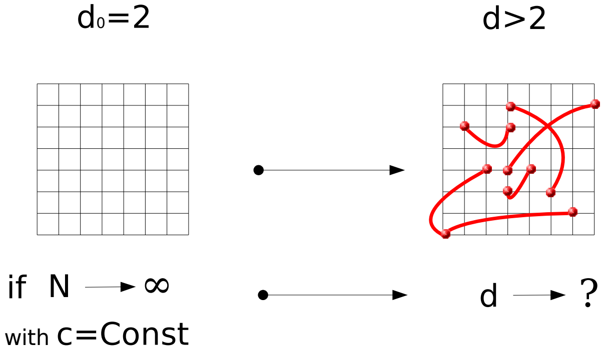
- Given an arbitrary graph $(\mathcal{L}_0, \Gamma_0)$ with N sites,
- We add $L = \frac{c \times N}{2}$ bonds uniformly over $\mathcal{L}_0$
- $\Rightarrow$ added connectivity Poisson distributed with mean c.

Poissonian Small-World Networks

- Given an arbitrary graph $(\mathcal{L}_0, \Gamma_0)$ with N sites,
- We add $L = \frac{c \times N}{2}$ bonds uniformly over $\mathcal{L}_0$
- $\Rightarrow$ added connectivity Poisson distributed with mean c.

Example $(\mathcal{L}_0, \Gamma_0)$ = Square Lattice

$N = 64$, $L = 6 \Rightarrow c = 12/64 \sim 0.19$



$d_0=2$

$d>2$

if $N \longrightarrow \infty$

with c=Const

$\bullet\!\longrightarrow$

$d \longrightarrow ?$

Basic properties of a Small-World Network

- for any finite $c > 0$ effectively $d(N) \rightarrow \infty$ and we have a short-distance behavior $l(N) \sim \log(N)$, as in random networks,
- large clustering coefficient, $C(N) \sim O(1)$, as in regular lattices.
- $\Rightarrow$ Small-World Networks represent an interesting interplay between finite and infinite dimensional networks!

Definition of the Model

$H_0 \to H_{\boldsymbol{c}}$

- Let us consider an Ising Model defined over $(\mathcal{L}_0, \Gamma_0)$

$$H_0 \equiv -J_0 \sum_{(i,j)\in\Gamma_0} \sigma_i\sigma_j - h \sum_{i\in\mathcal{L}_0} \sigma_i, \quad \text{The Pure Model,} \qquad (1)$$

- given any realization $\boldsymbol{c}$ of the small-world graph whose random bonds are determined by the adjacency matrix elements $c_{i,j} = 0, 1$, we define the corresponding small-world model

$$H_{\boldsymbol{c},\boldsymbol{J}} \equiv H_0 - \sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j, \quad \text{The Small} - \text{World Model} \qquad (2)$$

Definition of the Model

$H_0 \to H_{\boldsymbol{c}}$

- Let us consider an Ising Model defined over $(\mathcal{L}_0, \Gamma_0)$

$$H_0 \equiv -J_0 \sum_{(i,j)\in\Gamma_0} \sigma_i\sigma_j - h\sum_{i\in\mathcal{L}_0} \sigma_i, \quad \text{The Pure Model}, \qquad (1)$$

- given any realization $\boldsymbol{c}$ of the small-world graph whose random bonds are determined by the adjacency matrix elements $c_{i,j} = 0, 1$, we define the corresponding small-world model

$$H_{\boldsymbol{c},\boldsymbol{J}} \equiv H_0 - \sum_{i<j} c_{ij}J_{ij}\sigma_i\sigma_j, \quad \text{The Small} - \text{World Model} \qquad (2)$$

We want

F phase

- the magnetizations (F Order Parameter)

$$m^{(\mathrm{F})} \equiv \overline{\langle \sigma_i \rangle} \equiv \sum_{\boldsymbol{c}} P(\boldsymbol{c}) \int d\mathcal{P}\left(\boldsymbol{J}\right) \langle \sigma_i \rangle_{\boldsymbol{c},\boldsymbol{J}}, \tag{3}$$

- and the correlation functions

$$C^{(\mathrm{F})} \equiv \overline{\langle \sigma_i \sigma_j \rangle} \equiv \sum_{\boldsymbol{c}} P(\boldsymbol{c}) \int d\mathcal{P}\left(\boldsymbol{J}\right) \langle \sigma_i \sigma_j \rangle_{\boldsymbol{c},\boldsymbol{J}}, \tag{4}$$

- where $\langle \cdot \rangle_{\boldsymbol{c},\boldsymbol{J}}$ is the Boltzmann-average with the quenched graph $\boldsymbol{c}$ and the quenched couplings $\boldsymbol{J}$.
  $\boldsymbol{c}$ and $\boldsymbol{J}$ distributed according to $P(\boldsymbol{c})$ and $d\mathcal{P}\left(\boldsymbol{J}\right)$ where ...

where

$$d\mathcal{P}\left(\boldsymbol{J}\right) \equiv \prod_{(i,j),i<j} d\mu\left(J_{i,j}\right), \quad \int d\mu\left(J_{i,j}\right) = 1, \tag{5}$$

with

$$d\mu = \text{arbitrary measure} \tag{6}$$

$$P(\boldsymbol{c}) \equiv \prod_{(i,j),i<j} p(c_{i,j}), \quad \sum_{c_{i,j}=0,1} p(c_{i,j}) = 1, \tag{7}$$

with

$$p(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + \left(1 - \frac{c}{N}\right)\delta_{c_{ij},0}. \tag{8}$$

Similarly we want

SG phase

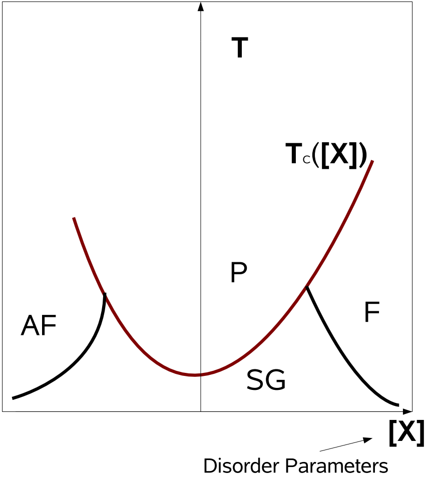
- the quadratic “magnetizations” (SG Order Parameter)

$$m^{(\mathrm{SG})} \equiv \overline{\langle \sigma_i \rangle^2} \equiv \sum_{\boldsymbol{c}} P(\boldsymbol{c}) \int d\mathcal{P}\left(\boldsymbol{J}\right) \langle \sigma_i \rangle^2_{\boldsymbol{c},\boldsymbol{J}}, \tag{9}$$

- the quadratic correlation functions

$$C^{(\mathrm{SG})} \equiv \overline{\langle \sigma_i \sigma_j \rangle^2} \equiv \sum_{\boldsymbol{c}} P(\boldsymbol{c}) \int d\mathcal{P}\left(\boldsymbol{J}\right) \langle \sigma_i \sigma \rangle^2_{\boldsymbol{c},\boldsymbol{J}}, \tag{10}$$

In Disordered Systems typically one expects . . .



The Cavity and the Replica Methods

- Cavity and Replica Methods (Parisi, Mezard, Zecchina, ...) are the most important analytical tools to analyze disordered models, especially when frustration is important
- These methods allow in particular to investigate the system exactly at temperatures lower than T_c
- However, a necessary condition to apply the Caivity and Replica Methods is that the graph be tree-like; we are not able to apply them when loops are present. Perturbative approaches exist (Chertkov and Chernyak), but in our case when $d_0 \geq 2$ loops are massively present! No way to face the model with these methods!

The Cavity and the Replica Methods

- Cavity and Replica Methods (Parisi, Mezard, Zecchina, ...) are the most important analytical tools to analyze disordered models, especially when frustration is important
- These methods allow in particular to investigate the system exactly at temperatures lower than T_c
- However, a necessary condition to apply the Caivity and Replica Methods is that the graph be tree-like; we are not able to apply them when loops are present. Perturbative approaches exist (Chertkov and Chernyak), but in our case when $d_0 \geq 2$ loops are massively present! No way to face the model with these methods!

The Cavity and the Replica Methods

- Cavity and Replica Methods (Parisi, Mezard, Zecchina, ...) are the most important analytical tools to analyze disordered models, especially when frustration is important
- These methods allow in particular to investigate the system exactly at temperatures lower than T_c
- However, a necessary condition to apply the Caivity and Replica Methods is that the graph be tree-like; we are not able to apply them when loops are present. Perturbative approaches exist (Chertkov and Chernyak), but in our case when $d_0 \geq 2$ loops are massively present! No way to face the model with these methods!

An Effective Field Theory

The Self-Consistent Equation

- Suppose to know m_0 of H_0 as a function of βJ_0 and βh: $m_0(\beta J_0, \beta h)$
- make the substitution: $h \to J^{(\mathrm{F})} m^{(\mathrm{F})} + h$

$$m_0(\beta J_0, \beta J^{(\mathrm{F})} m^{(\mathrm{F})} + \beta h), \tag{11}$$

where the Effective Coupling $\beta J^{(\mathrm{F})}$ is

$$\beta J^{(\mathrm{F})} = c \int d\mu(J) \tanh(\beta J), \tag{12}$$

An Effective Field Theory

The Self-Consistent Equation

- Suppose to know m_0 of H_0 as a function of βJ_0 and βh: $m_0(\beta J_0, \beta h)$
- make the substitution: $h \to J^{(\mathrm{F})} m^{(\mathrm{F})} + h$

$$m_0(\beta J_0, \beta J^{(\mathrm{F})} m^{(\mathrm{F})} + \beta h), \tag{11}$$

where the Effective Coupling $\beta J^{(\mathrm{F})}$ is

$$\beta J^{(\mathrm{F})} = c \int d\mu(J) \tanh(\beta J), \tag{12}$$

An Effective Field Theory

Order Parameters and T_c

- And now consider the following Self-Consistent Equation

$$m^{(\mathrm{F})} = m_0(\beta J_0, \beta J^{(\mathrm{F})} m^{(\mathrm{F})} + \beta h). \tag{13}$$

- It is possible to show that Eq. (13) is exact in P and on its boundary ⇒
- We have an exact way to calculate T_c

An Effective Field Theory

Order Parameters and T_c

- And now consider the following Self-Consistent Equation

$$m^{(\mathrm{F})} = m_0(\beta J_0, \beta J^{(\mathrm{F})} m^{(\mathrm{F})} + \beta h). \tag{13}$$

- It is possible to show that Eq. (13) is exact in P and on its boundary ⇒
- We have an exact way to calculate T_c

An Effective Field Theory

Order Parameters and T_c

- And now consider the following Self-Consistent Equation

$$m^{(\mathrm{F})}=m_0(\beta J_0,\beta J^{(\mathrm{F})}m^{(\mathrm{F})}+\beta h). \tag{13}$$

- It is possible to show that Eq. (13) is exact in P and on its boundary ⇒
- We have an exact way to calculate T_c

An Effective Field Theory

Correlation functions

- Once we know $m^{(\mathrm{F})}$, if we know $C_0(\beta J_0, \beta h) = \langle \sigma_i \sigma_j \rangle_0$ we have (rule: $h \to J^{(\mathrm{F})} m^{(\mathrm{F})} + h$)

$$C^{(\mathrm{F})} = C_0(\beta J_0, \beta J^{(\mathrm{F})} m^{(\mathrm{F})} + \beta h) + O\left(\frac{1}{N}\right) \qquad (14)$$

- It is possible to show that Eq. (14) is exact in P and on its boundary $\Rightarrow$
- Since in P $m^{(\mathrm{F})} = 0$ we know exactly that for $N \to \infty$ and $h = 0$ we have $C^{(\mathrm{F})} \to C_0(\beta J_0, 0)$:
  $\Rightarrow$ in the P region Pure Model and Small-World Model are indistinguishable!

An Effective Field Theory

Correlation functions

- Once we know $m^{(\mathrm{F})}$, if we know $C_0(\beta J_0, \beta h) = \langle \sigma_i \sigma_j \rangle_0$ we have (rule: $h \to J^{(\mathrm{F})} m^{(\mathrm{F})} + h$)

$$C^{(\mathrm{F})} = C_0(\beta J_0, \beta J^{(\mathrm{F})} m^{(\mathrm{F})} + \beta h) + O\left(\frac{1}{N}\right) \qquad (14)$$

- It is possible to show that Eq. (14) is exact in P and on its boundary $\Rightarrow$
- Since in P $m^{(\mathrm{F})} = 0$ we know exactly that for $N \to \infty$ and $h = 0$ we have $C^{(\mathrm{F})} \to C_0(\beta J_0, 0)$:
  $\Rightarrow$ in the P region Pure Model and Small-World Model are indistinguishable!

An Effective Field Theory

Correlation functions

- Once we know $m^{(\mathrm{F})}$, if we know $C_0(\beta J_0, \beta h) = \langle \sigma_i \sigma_j \rangle_0$ we have (rule: $h \to J^{(\mathrm{F})} m^{(\mathrm{F})} + h$)

$$C^{(\mathrm{F})} = C_0(\beta J_0, \beta J^{(\mathrm{F})} m^{(\mathrm{F})} + \beta h) + O\left(\frac{1}{N}\right) \qquad (14)$$

- It is possible to show that Eq. (14) is exact in P and on its boundary $\Rightarrow$
- Since in P $m^{(\mathrm{F})} = 0$ we know exactly that for $N \to \infty$ and $h = 0$ we have $C^{(\mathrm{F})} \to C_0(\beta J_0, 0)$:
  $\Rightarrow$ in the P region Pure Model and Small-World Model are indistinguishable!

An Effective Field Theory

Level of accuracy

- If frustration is not too high, the theory is exact at any T in the limit $c \to 0^+$ and $c \to \infty$.
- In any case the theory provides sensible estimations for magnetizations and the correlation functions at any T.
- Considering that we know exactly where the order parameters are not zero this is not a little result!

$d_0 = 0$, The Simplest Example: The Viana-Bray model

- Here $J_0 = 0 \Rightarrow$ this model can be seen as a $d_0 = 0$ small-world model.
- The solution of the Pure Model is very simple

$$m_0(0, \beta h) = \tanh(\beta h), \tag{15}$$

- $\Rightarrow$ the s.-c. Eq. gives (recall the rule $h \to J^{(\mathrm{F})} m^{(\mathrm{F})} + h$)

$$m^{(\mathrm{F})} \quad = \quad \tanh\left[\beta J^{(\mathrm{F})} m^{(\mathrm{F})} + \beta h\right], \tag{16}$$

- $\Rightarrow$ Classical Mean-Field critical behavior
- In particular from Eq. (16) it follows the critical surface (stability analysis with $h = 0$)

$$\beta_c^{(\mathrm{F})} J^{(\mathrm{F})} = 1, \quad \text{or, explicitly}: c \int d\mu(J) \tanh(\beta_c^{(\mathrm{F})} J) = 1.(17)$$

The One Dimensional Chain

Self-consistent equation

- Here $d_0 = 1$, the solution of the Pure Model still simple

$$m_0(\beta J_0, \beta h) = \frac{e^{\beta J_0} \sinh(\beta h)}{\left[e^{2\beta J_0} \sinh^2(\beta h) + e^{-2\beta J_0}\right]^{\frac{1}{2}}}, \qquad (18)$$

- then for the s.-c. equation we get ($h \to J^{(\mathrm{F})} m^{(\mathrm{F})} + h$)

$$m^{(\mathrm{F})} = \frac{e^{\beta J_0} \sinh(\beta J^{(\mathrm{F})} m^{(\mathrm{F})} + \beta h)}{\left[e^{2\beta J_0} \sinh^2(\beta J^{(\mathrm{F})} m^{(\mathrm{F})} + \beta h) + e^{-2\beta J_0}\right]^{\frac{1}{2}}}. \qquad (19)$$

The One Dimensional Chain

Stability Analysis

- If we try to develop for $h = 0$ and small field we find that the solution $m^{(\mathrm{F})} = 0$ is stable if

$$e^{2\beta^{(\mathrm{F})} J_0} \beta^{(\mathrm{F})} J^{(\mathrm{F})} < 1, \tag{20}$$

- and we find again a mean-field critical behavior with a critical temperature given by

$$e^{2\beta_c^{(\mathrm{F})} J_0} \beta_c^{(\mathrm{F})} J^{(\mathrm{F})} = 1, \quad \mathrm{or:}\ e^{2\beta_c^{(\mathrm{F})} J_0} c \int d\mu(J) \tanh(\beta_c^{(\mathrm{F})} J) = 1. \tag{21}$$

- If $J_0 \geq 0 \Rightarrow$ Eq. (21) has one and only one solution, but
- if $J_0 < 0 \Rightarrow$ Eq. (21) has either no solution or two solutions!

The One Dimensional Chain

Stability Analysis

- If we try to develop for $h = 0$ and small field we find that the solution $m^{(\mathrm{F})} = 0$ is stable if

$$e^{2\beta^{(\mathrm{F})} J_0} \beta^{(\mathrm{F})} J^{(\mathrm{F})} < 1, \tag{20}$$

- and we find again a mean-field critical behavior with a critical temperature given by

$$e^{2\beta_c^{(\mathrm{F})} J_0} \beta_c^{(\mathrm{F})} J^{(\mathrm{F})} = 1, \quad \mathrm{or}: e^{2\beta_c^{(\mathrm{F})} J_0} c \int d\mu(J) \tanh(\beta_c^{(\mathrm{F})} J) = 1. \tag{21}$$

- If $J_0 \geq 0 \Rightarrow$ Eq. (21) has one and only one solution, but
- if $J_0 < 0 \Rightarrow$ Eq. (21) has either no solution or two solutions!

The One Dimensional Chain

More stable solutions: Stability analysis is not enough!

- Actually, when $J_0 < 0$, the situation is a bit more complicated.
- In fact, when $J_0 < 0$, the self-consistent equation may admit MORE different and STABLE solutions
- $\Rightarrow$ It is not enough to study the stability of the solutions by using the self-consistent equation; we need also their effective free energy $L[m^{(\mathrm{F})}]$
- Among all the stable solutions we select the leading solution as the one that minimizes L.

The One Dimensional Chain: numerical examples

- Let’s see some examples with the choice:

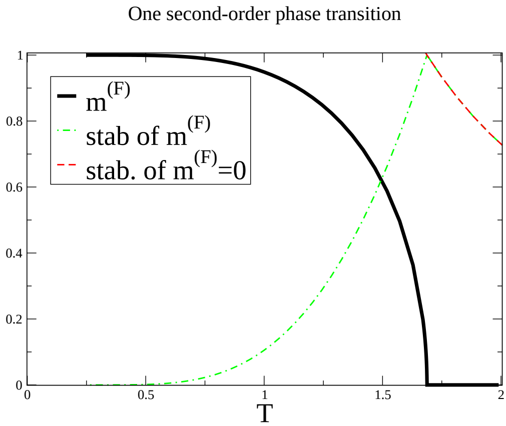
$$\frac{d\mu\left(J_{i,j}\right)}{dJ_{i,j}} = \delta\left(J_{i,j} - J\right). \tag{22}$$

- We plot (in black) the leading solution $m^{(\mathrm{F})}$,
- we also plot (in red) the curve for the stability of $m^{(\mathrm{F})} = 0$

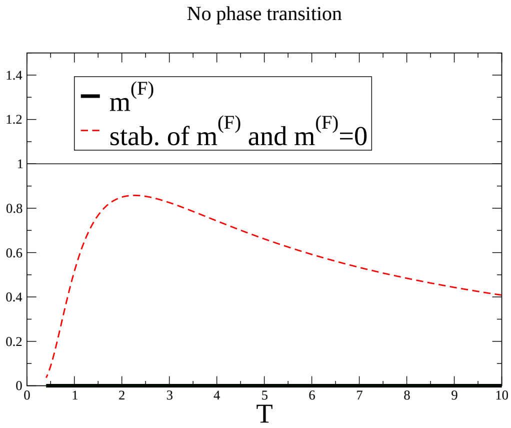
$$y = e^{2\beta_c^{\mathrm{F}} J_0} \beta_c^{(\mathrm{F})} J^{(\mathrm{F})}, \tag{23}$$

- and a similar curve (in green) for the stability of the leading solution $m^{(\mathrm{F})}$

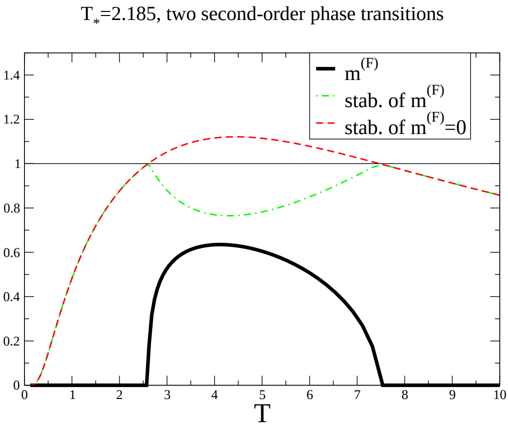
The One Dim. Chain, $J_0 = 1$, $J = 3/5$, $c = 0.5$



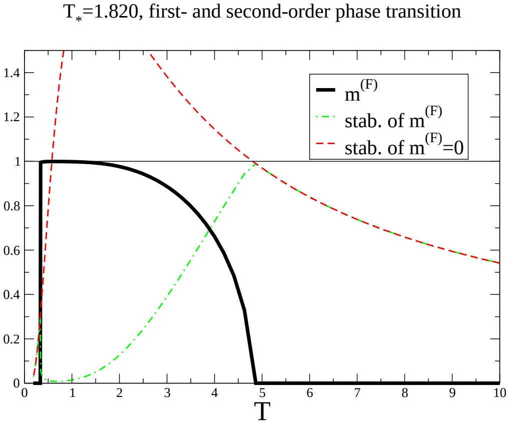
The One Dim. Chain, $J_0 = -1$, $J = 1$, $c = 5$



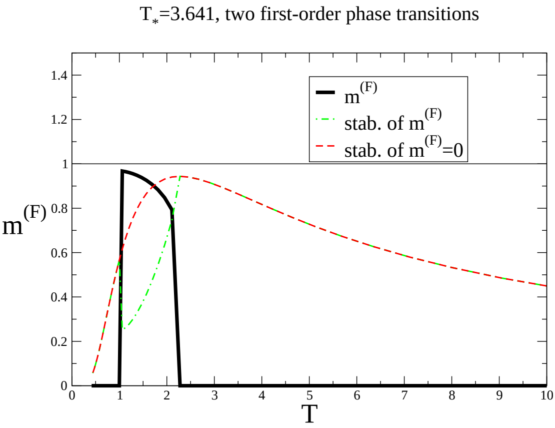
The One Dim. Chain, $J_0 = -0.6$, $J = 7$, $c = 1.6$



The One Dim. Chain, $J_0 = -0.5$, $J = 1$, $c = 6$



The One Dimensional Chain, $J_0 = -1$, $J = 1$, $c = 5.5$



The Dual Phase Transition Scenario

Landau Analysis

- By expanding $L[m^{(\mathrm{F})}]$ in power of $m^{(\mathrm{F})}$ is possible to show that the following scenario holds in general for any graph $(\mathcal{L}_0, \Gamma_0)$ with any d_0!
- If $J_0 \geq 0 \Rightarrow$ for any c we have one - and only one - second-order phase transition with the mean-field classical critical indices, but with a finite correlation length
- If $J_0 < 0 \Rightarrow$ there is some c_{min} below which there is no transition and over which we have at least two T_c corresponding to first- or second-order phase transitions, depending on whether we have one or more stable solutions $\Rightarrow$ complex models with many solutions will likely have many first-order phase transitions.

Summary

- We have seen that a very simple and general Effective Field Theory is capable to deal with Loops!
- If we can solve the Pure Model, analytically or numerically, we get the solution of the Small-World Model
- The power of this method has allowed us to discover a very general and unexpected result: when $J_0 < 0$ we have Multicritical Points with First- and Second-Order Phase Transitions.

References

- M. Ostilli and J. F. F. Mendes, arXiv:0801.3454
- M. Ostilli and J. F. F. Mendes, arXiv:0801.3563