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Scale-Free Directed Networks from Threshold Model with Two Intrinsic Node Variables

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Scale-Free Network Model

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- Growing network model
  - Preferential attachment: likelihood of receiving new links is proportional to the degrees of nodes
  - Examples: Barabasi-Albert (BA) model

  (Many variations)

- Non-growing network model
  - Intrinsic variable: property of each node
  - Two nodes are connected if their intrinsic variables meet a certain condition
  - Examples: (Vertex intrinsic) Fitness model

    Threshold model

Fitness Model

- Each node has an intrinsic variable (fitness).

  x_i: fitness of node i

- An undirected link between node i and node j is established with probability $f(x_i, x_j)$

  An example: $$f(x_i, x_j) = \frac{x_i x_j}{(x_M)^2}$$

- Power-law distributed fitness produces scale-free networks.

Threshold Model

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- Each node has an intrinsic variable (weight).

  W_i : weight of node i

- An undirected link between node *i* and node *j* is established if and only if

  $$W_i + W_j \geq \theta \qquad \theta \text{ : threshold}$$

- Exponentially distributed weight produces scale-free networks.

Properties of Fitness and Threshold Models

Network Model	Degree Distribution		Clustering Coefficient – Degree relationship	
	Power-Law	Scaling Exponent	Power-Law	Scaling Exponent
Fitness Model (Power-law fitness)	Exist	arbitrary	Non-exist	
Threshold Model (exponential weight)	Exist	3	Exist	2
Real Networks	Exist	2 - 3	Exist	0 - 1

Aim of the Study

- Focusing on the threshold model
  - Because it is highly analytically tractable
- Extension of the threshold model in order to have
  - More variability of the scaling exponents of degree distribution and clustering coefficient
  - Applicability to directed networks

Extension of the Threshold Model (1)

- Each node has two weights:

  W_i^{out} : outgoing weight (the tendency to have outgoing links)

  W_i^{in} : incoming weight (the tendency to have incoming links)

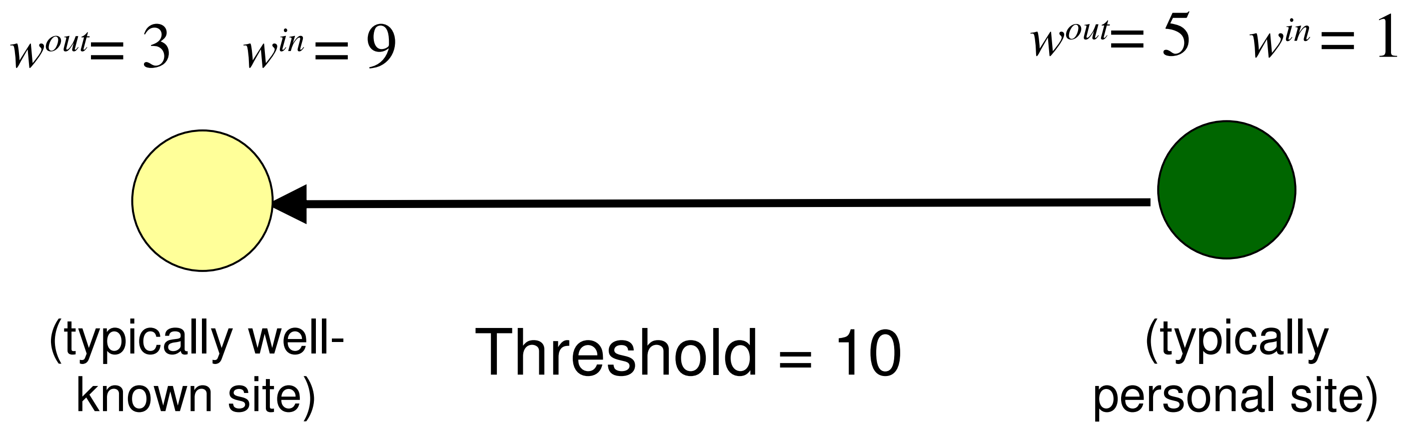
- A directed link from node i to node j is established if and only if

$$W_i^{out} + W_j^{in} \geq \theta$$

θ : threshold

Extension of the Threshold Model (2)

- Example



- Incoming and outgoing weights of a node are either independent or (positively) correlated.

Degree Distribution (1)

- The proposed model is highly analytical tractable.
- Degree distribution and clustering coefficient have explicit representation.

$$P_c^{in}(n) = G_c(\theta - F_c^{-1}(n/N))$$ In-degree distribution

$$P_c^{out}(n) = F_c(\theta - G_c^{-1}(n/N))$$ Out-degree distribution

$F_c(x)$: Survival function of outgoing weight

$G_c(x)$: Survival function of incoming weight

Degree Distribution (2)

- Exponentially distributed weights produces scale free networks.
  - Out-degree distribution $P_c^{out}(n) = Ne^{-\lambda_1} n^{-\lambda_1/\lambda_2}$
  - In-degree distribution $P_c^{in}(n) = Ne^{-\lambda_2} n^{-\lambda_2/\lambda_1}$
- Scaling exponents of out-degree and in-degree distributions are controllable in the range (1, ∞)

$$\gamma_d^{out} = (\lambda_1 + \lambda_2)/\lambda_2 \qquad \gamma_d^{in} = (\lambda_1 + \lambda_2)/\lambda_1$$

λ_1^{-1}: Average outgoing weight λ_2^{-1}: Average incoming weight

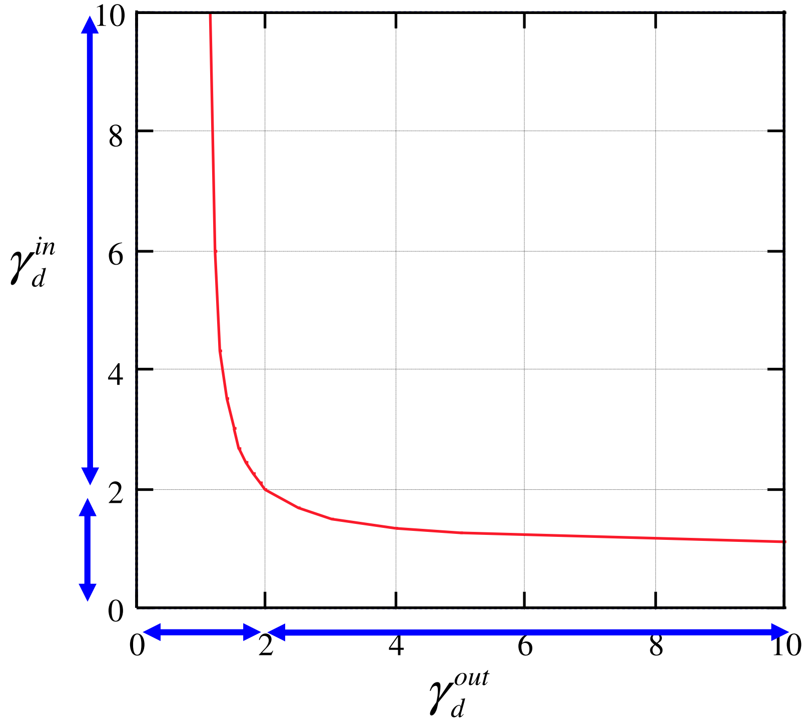
Degree Distribution (3)

- Both scaling exponents are closely related

$$\gamma_d^{out} = \frac{\gamma_d^{in}}{\gamma_d^{in} - 1}$$

γ_d^{out} Scaling exponent of out-degree

γ_d^{in} Scaling exponent of in-degree



Clustering Coefficient (1)

- Clustering coefficient – degree relationship exhibits power-law behavior (exponential weights)
  - Incoming and outgoing weights are independent

$$C_{out}(n) = \begin{cases} \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 \theta} \frac{N}{n} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 \theta} \left(\frac{N}{n} \right)^{\lambda_2 / \lambda_1} & \lambda_1 \neq \lambda_2 \\ e^{-\lambda_1 \theta} \frac{N}{n} \left(1 + \log \frac{n}{N} + \lambda_1 \theta \right) & \lambda_1 = \lambda_2 \end{cases}$$

  - Incoming and outgoing weights are strongly correlated

$$C_{out}(n) = \begin{cases} \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 \theta} \left(\frac{N}{n} \right)^{\frac{\lambda_1 + \lambda_2}{\lambda_2}} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 \theta} \left(\frac{N}{n} \right)^{\frac{\lambda_1 + \lambda_2}{\lambda_1}} & \lambda_1 \neq \lambda_2 \\ e^{-\lambda_1 \theta} \frac{N^2}{n^2} \left(1 + 2 \log \frac{n}{N} + \lambda_1 \theta \right) & \lambda_1 = \lambda_2 \end{cases}$$

Clustering Coefficient (2)

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- Scaling exponents depends on the correlation between outgoing and incoming weight
  - Scaling exponent < 1 if both weights are independent
  - Scaling exponent is in (1, 2) if both weights are positively correlated

Comparison with Original Threshold Model

Network Model	Degree Distribution		Clustering Coefficient – Degree Relation	
	Power-Law	Scaling Exponent	Power-Law	Scaling Exponent
Proposed Model	Exist	1 - ∞	Exist	0 - 2
Original Threshold Model	Exist	3	Exist	2
Real Networks	Exist	2 - 3	Exist	0 - 1

Simulation Condition

- Number of nodes = 50000
- Threshold = 10
- Incoming and outgoing weights are exponentially distributed
  - Average incoming weight was equal to 1.
  - Average outgoing weight was set at 0.5, 1.0 or 2.0.

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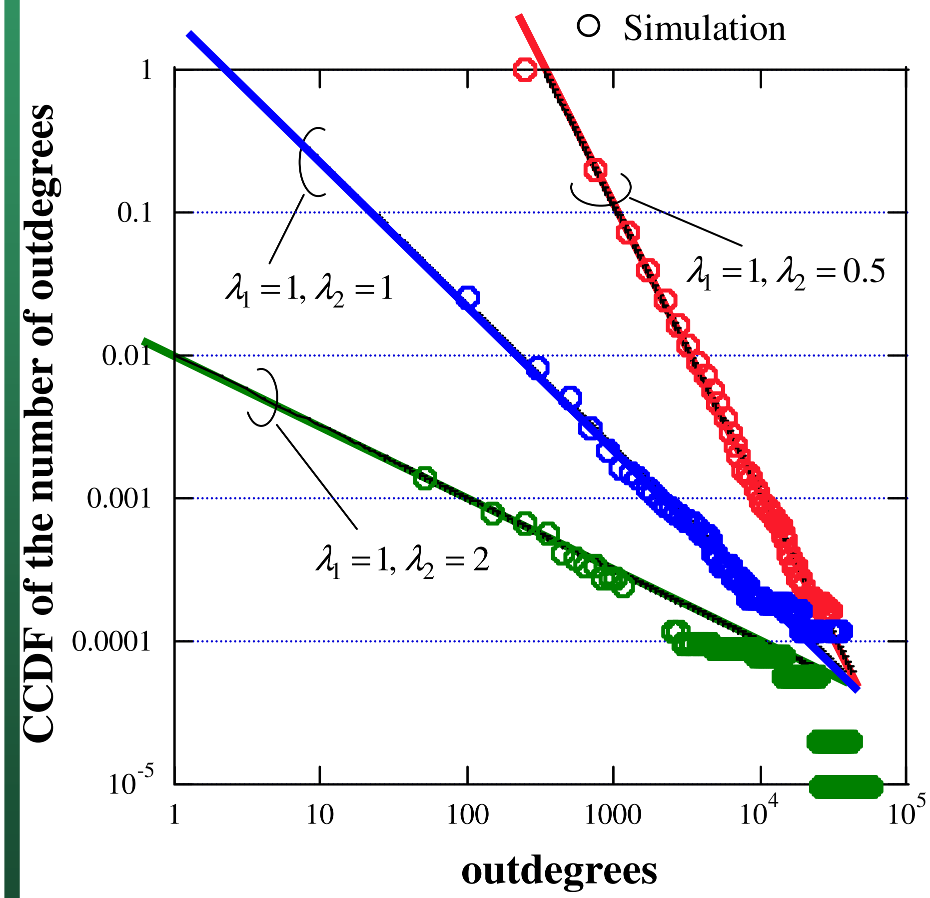
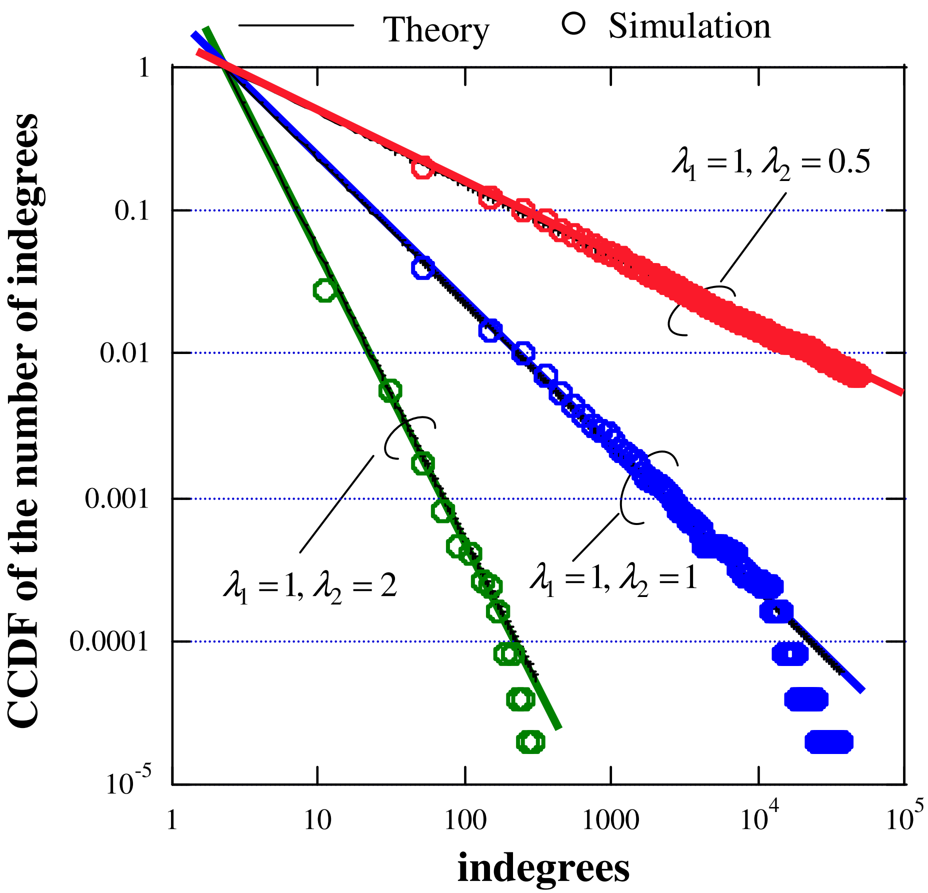
Indegree and Outdegree Distribution

Outdegree

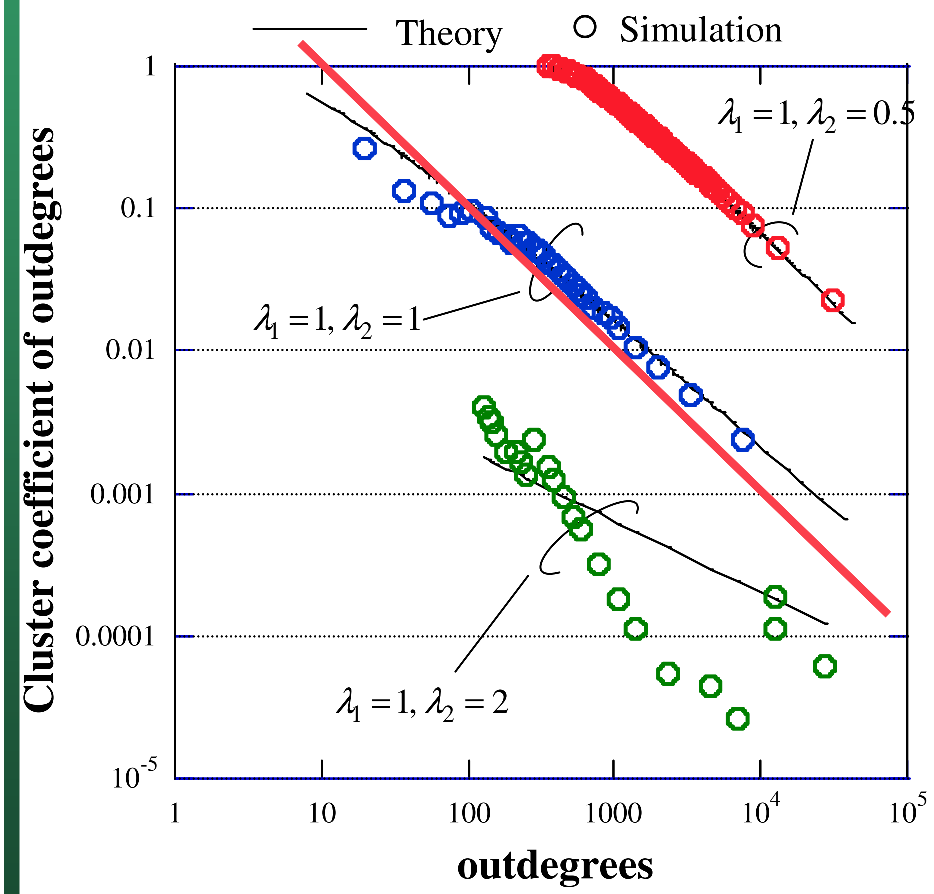


Indegree

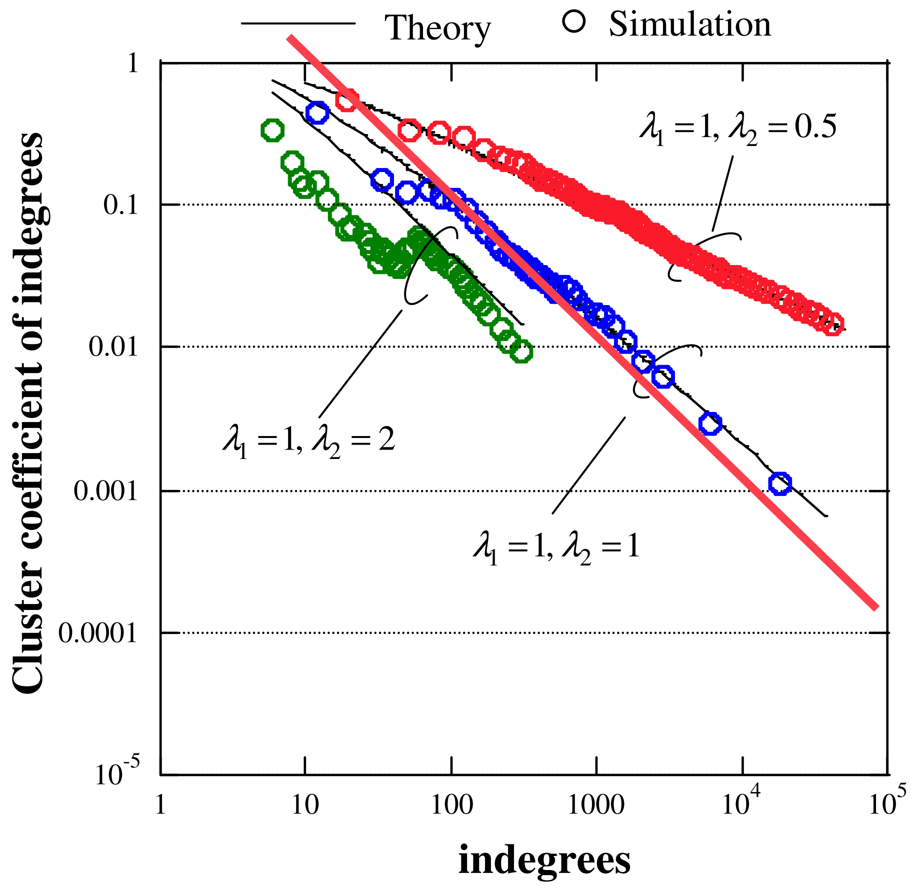


Clustering Coefficient (Independent)

Clustering coefficient of outdegree

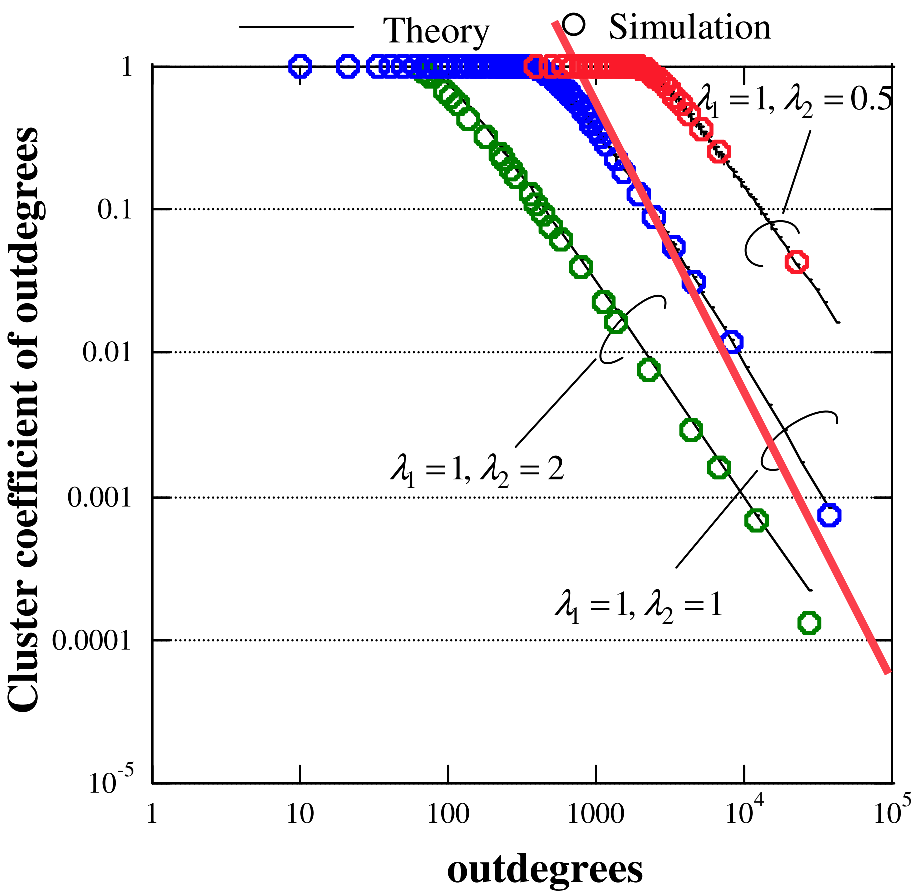


Clustering coefficient of indegree

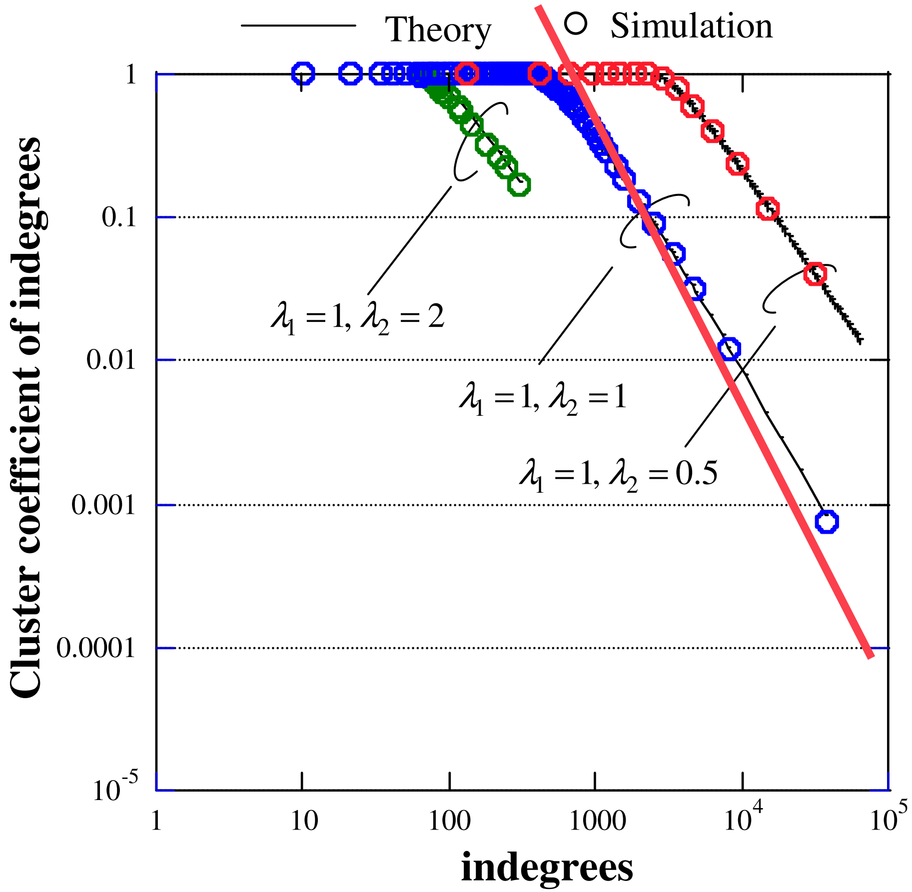


Clustering Coefficient (Correlated)

Clustering coefficient of outdegree

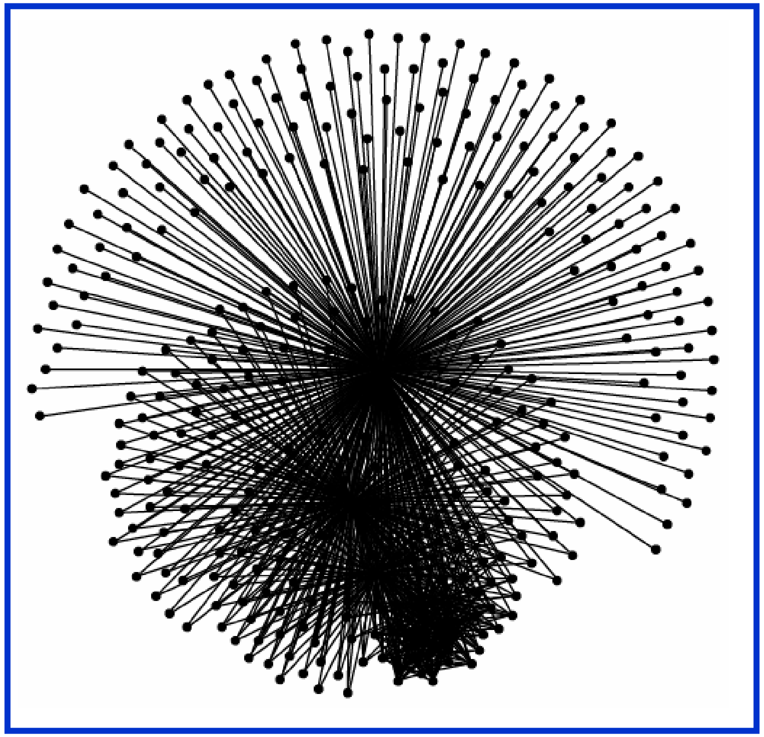


Clustering coefficient of indegree

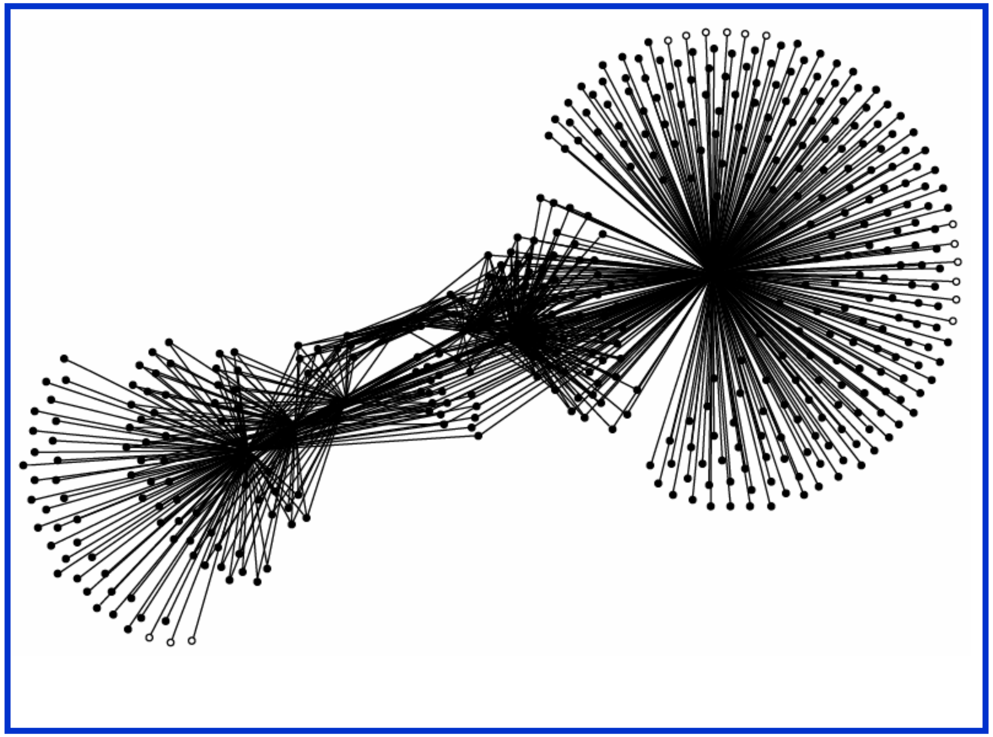


Independence of Weights

Closely Correlated Weights



Independent Weights



Conclusion

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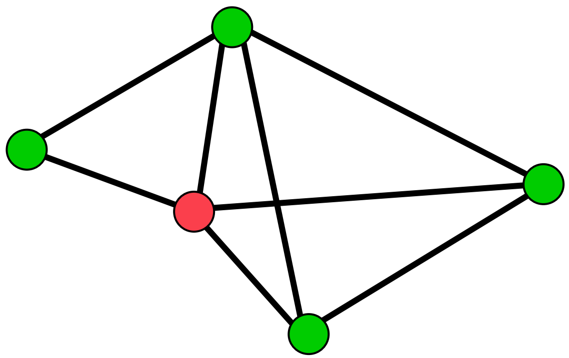
- We have proposed a simple extension of the threshold model
  - Applicability to directed networks
  - Variability of scaling exponents of degree distribution and clustering coefficients

- Further improvement of the model is required to have better agreement with real networks.
  - The relationship between scaling exponents of in-degree and out-degree distributions is too restrictive

Thank you

Clustering

- Examples



Clustering Coefficient

$$= \frac{4}{4 \times 3/2} = \frac{2}{3}$$

- Average clustering coefficient of nodes of degree n, $C(n)$, exhibits a power-law behavior.

$$C(n) \propto n^{-\gamma_c}$$

γ_c : scaling exponent

Scaling Exponent

Degree Distribution

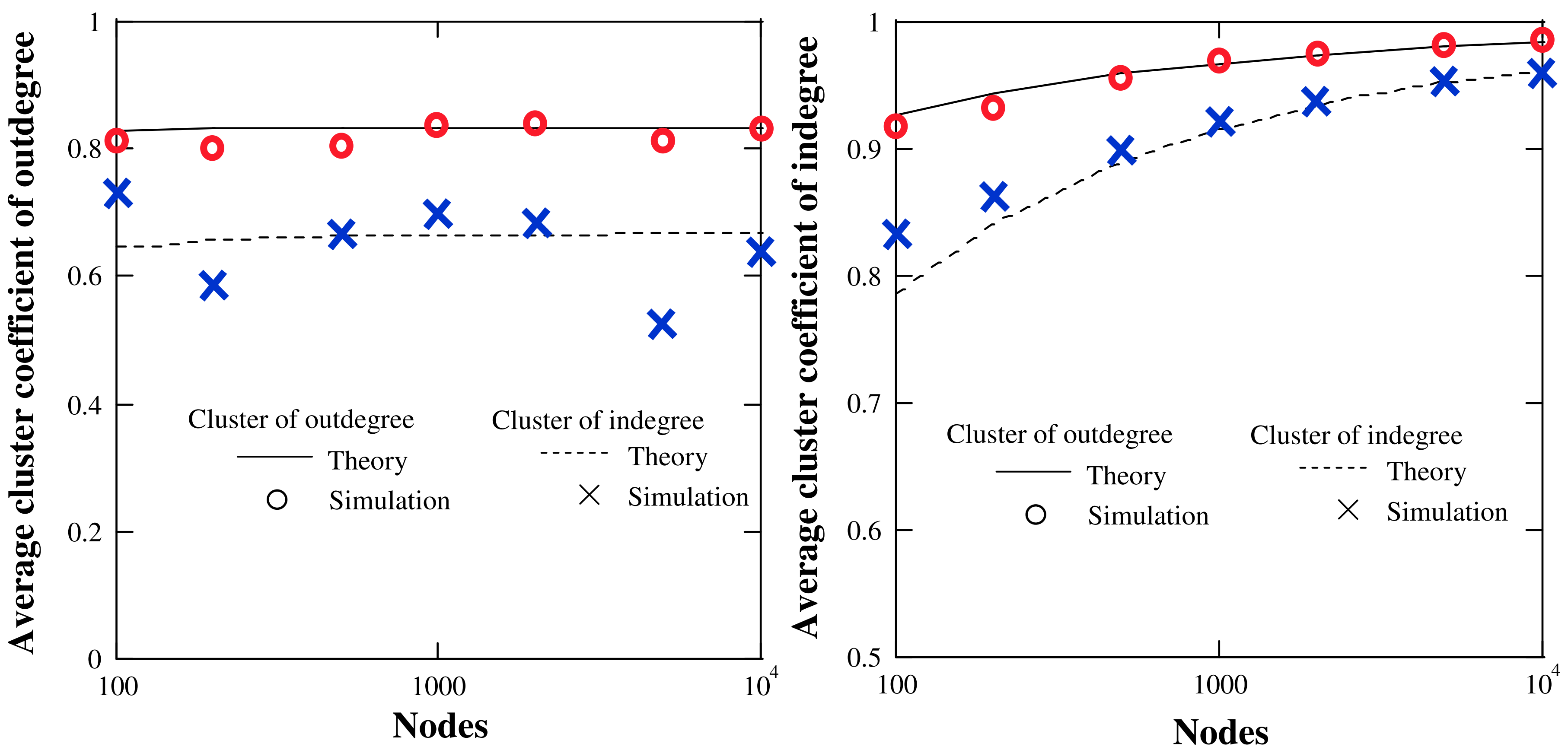
Network	γ_d
WWW	1.94 - 2.72
Movie actors	2.3
Language network	2.7～2.8
Internet AS level	2.1 - 2.2
Internet router level	2.2 - 2.5

Clustering Coefficient

Network	γ_c
WWW	1.0
Movie actors	1.0
Language network	1.0
Internet AS level	0.75
Internet router level	0

Average Clustering Coefficient

Independent **Correlated**



Extension to the undirected network modeling

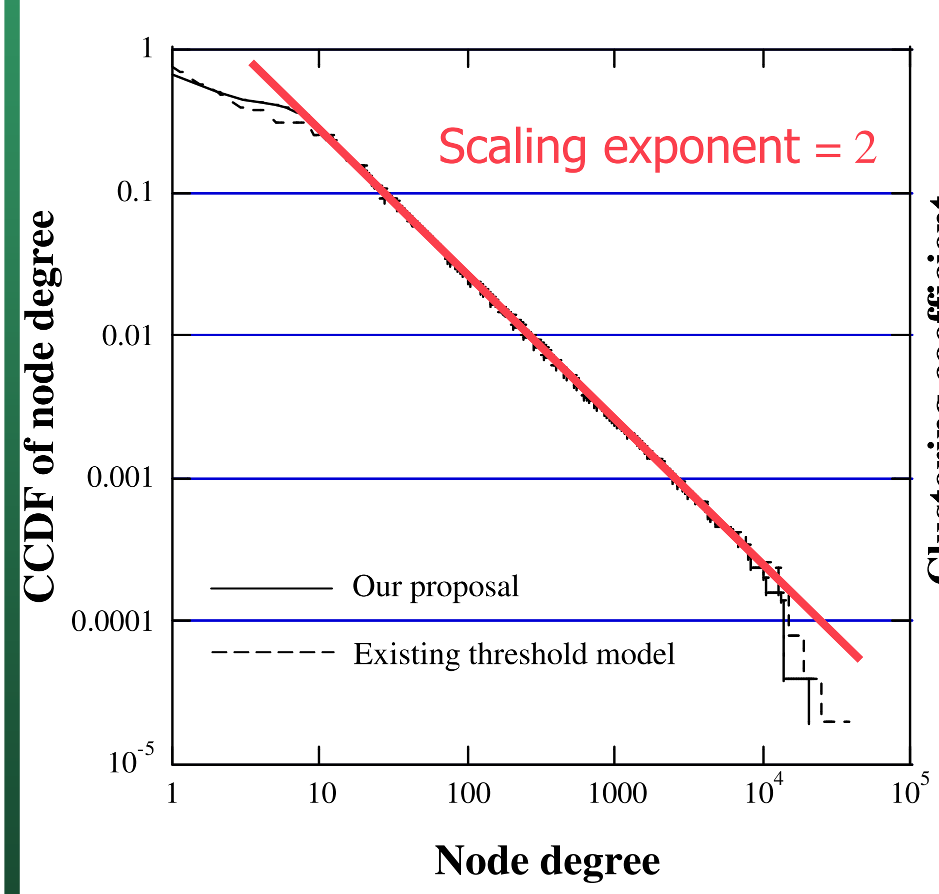
- Extension to the undirected network modeling is possible by modifying link establishing mechanism
  - Nodes i and j is connected if and only if

$$W_i^{out} + W_j^{in} \geq \theta \ \ or \ \ W_i^{in} + W_j^{out} \geq \theta$$

- If outgoing and incoming weights are independent, proposed model yields different results with the original threshold model.

Comparison with Original Threshold Model

Degree Distribution



Clustering Coefficient