Who wins a group-group \textcolor{red}{competition}?

Phase-transitions in Ising models of coupled complex networks

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Social interactions – often majority-driven
- Social interaction networks are often modular (subgroups)
There is often a **competition** between different subgroups
What is a fitness function for social groups competition?

- Size?
- Density of internal connections?
- Strength of internal links?
- Distribution of links?

How to measure it?
Introduction

- We represent simplest social interactions with Ising model on network
Scale-free networks

What does the weighted spin mean, and why weighted?

\[ k_1 = 6 \]
\[ k_2 = 2 \]

\[ s_1 + s_2 = 0 \]
no order?

\[ s_1 k_1 + s_2 k_2 < 0 \]
order!
Temperature and internetwork interaction strength have similar effect on antiparallel-parallel transition.

Networks become unseparable at internetwork density equal to intranetwork density.
Scale free B-A network wins a competition against a random E-R graph.

Net A : B-A, Net B : E-R

\[ N_A = N_B = 100, \ E = 100 \]

\[ \langle k_B \rangle = 20, \ \text{various} \ \langle k_A \rangle \]
Network A (red) is smaller, but denser. Network B (green) is larger, but sparse.

The internetwork connections are added with time.

The network with larger fluctuation loses.

Network with small “strength” suffers large fluctuations – fluctuating networks flip over first and therefore lose
Susceptibility — a measure how external field changes order parameter

$$\chi = \frac{\partial S_{A}}{\partial \lambda}$$

order parameter

interaction \( \lambda \)

Fluctuation-dissipation theorem

$$\chi = \frac{\partial S_{A}}{\partial \lambda} = \text{susceptibility} \sim \text{fluctuations} = <(S_{A} - <S_{A}>)^2>$$

small fluctuations \( \rightarrow \) WIN

because smaller fluctuations \( \rightarrow \) smaller susceptibility thus other network (with larger susceptibility) changes spin faster
Theory of coupled networks

System of two coupled networks (one larger modular network)
Coupled networks

$S_A$ - spin of network A weighted by $k_{AA}$ (●)

$S_{BA}$ - spin of network B weighted by $k_{BA}$ (○)
Coupled networks

The state of the system can be written as vector:

\[
\begin{pmatrix}
S_A \\
S_{AB} \\
S_{BA} \\
S_B
\end{pmatrix}
\]
Coupled networks

If we assume that the number of inter-network connections is proportional to the intra-network degree:

\[ k_{AB} = p_A k_{AA}, \quad k_{BA} = p_B k_{BB} \]

then we do not have to consider \( S_{AB} \) and \( S_{BA} \) anymore since they are proportional to \( S_A \) and \( S_B \).

\[
\begin{bmatrix}
\Lambda_{AA} & \Lambda_{BA} \\
\Lambda_{AB} & \Lambda_{BB}
\end{bmatrix}
\begin{pmatrix}
S_A \\
S_B
\end{pmatrix}
= \lambda
\begin{pmatrix}
S_A \\
S_B
\end{pmatrix}
\]
Coupled networks

We have following eigenvalues $\lambda$ of the matrix $\Lambda$:

$$\lambda_{\pm} = \frac{\Lambda_{AA} + \Lambda_{BB} \pm \sqrt{\left(\Lambda_{AA} - \Lambda_{BB}\right)^2 + 4\Lambda_{BA}\Lambda_{AB}}}{2}$$

$\lambda_- :$ eigenvector $\begin{pmatrix} 1 \\ -c_1 \end{pmatrix}$

$\lambda_+ :$ eigenvector $\begin{pmatrix} 1 \\ c_2 \end{pmatrix}$

The networks are ordered antiparallelly. $T_C$ is lower than for separate networks.

Increasing inter-network connection strengths causes $T_C$ to decrease, down to 0, when the inter-network connections are as dense as intra-network.

The networks are ordered parallelly. $T_C$ is higher than for separate networks.

Networks stabilize each other, similar to way the network stabilizes itself.
N_A = N_B = 100, \langle k_A \rangle = \langle k_B \rangle = 20

E = 100 (number of inter-network connections)

There are different scenarios by the same networks but different internetwork connections
\[N_A = N_B = 100\]
\[<k_A> = 20\]
\[<k_B> = 18\] network B is "weaker"

\[s(T)\]

**few connections**

\[\text{(20)}\]

**many connections**

\[\text{(100)}\]

Nec Hercules contra plures!
Conclusions

- Competition of two social groups can bring a discontinuous change in their opinions as a function of intergroup connections or noise level.
- Larger group is not always a winner!
- Density and distribution of internal links is also important.
- Winner can depend on the choice of group-group interactions.
- Observation of proper fluctuations before the competition can be a useful indicator.
The simplest confrontation: chain and star

\[ H_{chain} = -J \sum_{i=1}^{N} S_i S_{i+1} \quad H_{star} = -J S_0 \sum_{i=1}^{N} S_i \]
Statistical physics of a star network

\[ E_{\text{star}}^k = -J(1)k - J(-1)(N - k) \]

\[ Z_{\text{star}} = \sum_{s_0, \{s_i\} = -1,1} \exp(-\beta E_k) = 2\sum_{k=0}^{N} \binom{N}{k} \exp(\beta Jk) \exp[-\beta J(N - k)] \]

\[ Z_{\text{star}} = \ldots = 2(e^{-\beta J} + e^{+\beta J})^N = Z_{\text{chain}} \]
Star network in homogenous external magnetic field

\[ H_{\text{star}} = -JS_0 \sum_{i=1}^{N} S_i - B \sum_{i=0}^{N} S_i \]

\[ \langle s_0 s_i \rangle = \tanh(\beta J) \quad \langle s_j s_i \rangle = \tanh^2(\beta J) \]

\[ \chi_{\text{star}} = \frac{\partial \langle S \rangle}{\partial B} = \sum_{i,j=0}^{N} \langle s_i s_j \rangle = \beta \left[ (N + 1) + 2N \tanh(\beta J) + N(N - 1) \tanh^2(\beta J) \right] \]

\[ \chi_{\text{star}} / N \approx N\beta \tanh^2(\beta J) \quad \chi_{\text{chain}} / N \approx \beta \exp(2\beta J) \]
Conclusions

- Larger group is not always a winner!
- Density and distribution of internal links is also important
- Winner can depend on the choice of group-group interactions
- Observation of proper fluctuations before the competition can be a useful indicator.
First order phase transition in Ising model on two connected Barabasi-Albert networks
Curve of susceptibility balance between spin chain and spin star
preferential

A

B

hub

N_A = N_B = 100

<k_A> = 20

<k_B> = 18  network B is “weaker”

Nec Hercules contra plures!

S(T)

single simulation

averaged (100)

few connections

many connections

(20)

(100)
The simplest confrontation: chain and star

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Curve of susceptibility balance between spin chain and spin star
Statistical Physics of Hamiltonian Graphs

\[ S = - \sum_G P(G) \ln P(G), \]  

(1)

subject to the constraints

\[ \langle x_i \rangle = \sum_G x_i(G) P(G) \]  

(2)

for \( i = 1, 2, \ldots, r \), plus the normalization condition

\[ \sum_G P(G) = 1. \]  

(3)

The Lagrangian for the above problem is given by the expression

\[ \mathcal{L} = - \sum_G P(G) \ln P(G) + \alpha \left( 1 - \sum_G P(G) \right) \]  

(4)

\[ + \sum_{i=1}^{r} \theta_i \left( \langle x_i \rangle - \sum_G x_i(G) P(G) \right), \]  

(5)
\[ P(G) = \frac{e^{-H(G)}}{Z}, \]  

where \( H(G) \) is the network Hamiltonian

\[ H(G) = \sum_{i=1}^{r} \theta_i x_i(G), \]  

and \( Z \) represents the partition function (normalization constant)

\[ Z = \sum_{G} e^{-H(G)} = e^{\alpha+1}. \]
Fluctuation-dissipation relations for Hamiltonian graphs

\[ \langle x_i^2 \rangle - \langle x_i \rangle^2 = - \frac{\partial \langle x_i \rangle}{\partial \theta_i} = \chi_i^{(\theta)} \]

\[ \langle E^2 \rangle - \langle E \rangle^2 = - \frac{\partial E}{\partial \beta} = kT^2 C_V \]

\[ \chi_i^{(\theta)} = - \frac{\partial \langle k_i \rangle}{\partial \theta_i} = \langle k_i^2 \rangle - \langle k_i \rangle^2 = \sum_j p_{ij} (1 - p_{ij}) = \langle k_i \rangle - \sum_j p_{ij}^2 \]
Fluctuation-dissipation relations for Hamiltonian stars

$$\chi = - \frac{1}{\langle k \rangle} \frac{\partial \langle k \rangle}{\partial \theta} = 1 - \frac{(N - \langle k^* \rangle)^2}{N^2(N - 1)} - \frac{\langle k^* \rangle^2}{N^2}$$

for the bulk of nodes and

$$\chi^* = - \frac{1}{\langle k^* \rangle} \frac{\partial \langle k^* \rangle}{\partial \theta^*} = 1 - \frac{\langle k^* \rangle}{N}$$

for the supernode.

Zapraszamy do uczestnictwa w 3 Ogólnopolskim Sympozjum FENS 2007 poświęconym zastosowaniom metod i metodologii fizyki w szeroko rozumianej ekonomii, inżynierii finansowej, a także w naukach społecznych.


Celem Sympozjum jest:

1. Spotkanie fizyków, których tematyka badawcza obejmuje również obszary nauk ekonomicznych i społecznych.
2. Spotkanie fizyków z matematykami, ekonomistami, socjologami zainteresowanymi współpracą naukową oraz z praktykami finansów i biznesu (przedsiębiorcami banków, instytucji ubezpieczeniowych, firm komputerowych, giełdy, etc.).
3. Dyskusja nad problemami i potrzebami kształcenia studentów oraz doktorantów w zakresie ekono- i socjofizyki.

Scaling and Universality in Proportional Elections

Santo Fortunato\textsuperscript{1} and Claudio Castellano\textsuperscript{2}

FIG. 2 (color online). Universality of the scaling function $F(vQ/N)$ across different countries and years. The lognormal fit, performed on the Polish curve, describes very well the data. The universal curve is well reproduced by our model, where the dynamics of the voters’ opinions reflects the spreading of the word of mouth in the party’s electorate.

FIG. 3 (color online). Spreading of the word of mouth among voters. The candidate (right) convinces some of his or her contacts to vote for him/her. The convinced voters become “activists” and try to convince some of their acquaintances, and so on. Successful interactions are indicated by solid lines, unsuccessful interactions are displayed as dashed lines.
Regular networks

Hamiltonian for the Ising model, assuming the interaction constant is same for all pairs of spins.

\[ H = -J \sum_{i,j} s_is_j \]

Equation for mean spin (magnetization) in the mean-field approximation:

\[ \langle s_i \rangle = \tanh \left( \beta \sum_j J_{ij}s_j \right) = \tanh \left( \beta Jk \langle s_i \rangle \right) \]

k - coordination number (number of neighbors)
Scale-free networks

\[ \langle s_i \rangle = \tanh \left( \beta \sum_j J_{ij} s_j \right) = \tanh \left( \beta J \sum_j \varepsilon_{ij} \langle s_j \rangle \right) \]

Instead of coincidence matrix \( \varepsilon_{ij} \) we take the statistical probability of the spins being neighbors:

\[ \langle \varepsilon_{ij} \rangle = \frac{k_i k_j}{E} = \frac{k_i k_j}{\langle k \rangle N} \]

We obtain following

\[ \langle s_i \rangle = \tanh \left( \frac{\beta J k_i}{\langle k \rangle N} \sum_j k_j \langle s_j \rangle \right) \]
Scale-free networks

\[ \langle s_i \rangle = \tanh \left( \frac{\beta J k_i}{\langle k \rangle N} \sum_j k_j \langle s_j \rangle \right) \]

Now we define average weighted spin \( S \) as:

\[ S = \frac{1}{\langle k \rangle N} \sum_i k_i \langle s_i \rangle \]

so we can write above as

\[ S = \frac{1}{\langle k \rangle N} \sum_i k_i \tanh \left( \beta J k_i S \right) \]
Scale-free networks

\[ S = \frac{1}{\langle k \rangle N} \sum_i k_i \tanh(\beta J k_i S) \]

Linear approximation

\[ S = \frac{1}{\langle k \rangle N} \sum_i k_i^2 \beta J S = \beta J \frac{\sum k_i^2}{\langle k \rangle N} S = \beta J \frac{\langle k^2 \rangle}{\langle k \rangle} S \]

B-A network

\[ \beta J \frac{\langle k^2 \rangle}{\langle k \rangle} S = \beta J \frac{m}{2} \ln NS \quad \rightarrow \quad T_c = J \frac{m}{2} \ln N \]
Effective Tc versus m+ N for m = 5 and various N, averaged over up to 1000 samples.

Ferromagnetic phase transition in Barabási–Albert networks

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Coupled networks

Using linear approximation we investigate existence of nonzero solutions, that correspond to an ordered phase.

\[
\langle s_{Ai} \rangle = \beta J_{AA} k_{AAi} \sum_j \frac{k_{Aj} \langle s_{Aj} \rangle}{E_{AA}} + \beta J_{BA} k_{ABI} \sum_j \frac{k_{BAj} \langle s_{Bj} \rangle}{E_{BA}}
\]

\[
\langle s_{Bi} \rangle = \beta J_{BB} k_{BBi} \sum_j \frac{k_{Bj} \langle s_{Bj} \rangle}{E_{BB}} + \beta J_{AB} k_{BAi} \sum_j \frac{k_{ABj} \langle s_{Aj} \rangle}{E_{AB}}
\]

Similar to single scale-free network we introduce weighted spins \( S_A \), \( S_B \). Unfortunately we also need \( S_{AB} \) and \( S_{BA} \).
Numerical results

Numerical simulations for two same B-A (N=5000, <k>=10, D=0) and various number of inter-network connections:

$T_{c+}$, parallel ordering, $T_{c+} = A + C$

Symbols - numeric results

Lines - analytic formulas

$T_{c-}$, antiparallel ordering, $T_{c-} = A - C$

Inter-network link number
Numeric results

We have performed numeric simulations to find the critical temperatures $T_{c^+}$ and $T_{c^-}$.

To find the critical temperature $T_{c^+}$ we calculate numerically the susceptibility $\chi = S_h - S_0$. Due to very highly fluctuating nature we make 30-point running average and fit parabolic curve.
Numeric results

We have performed numeric simulations to find the critical temperatures $T_{c+}$ and $T_{c-}$.

Plot the total spin starting from antiparallel ordered state as a function of temperature.