

Who wins a group-group **competition** ?

Phase-transitions in Ising models of coupled complex networks

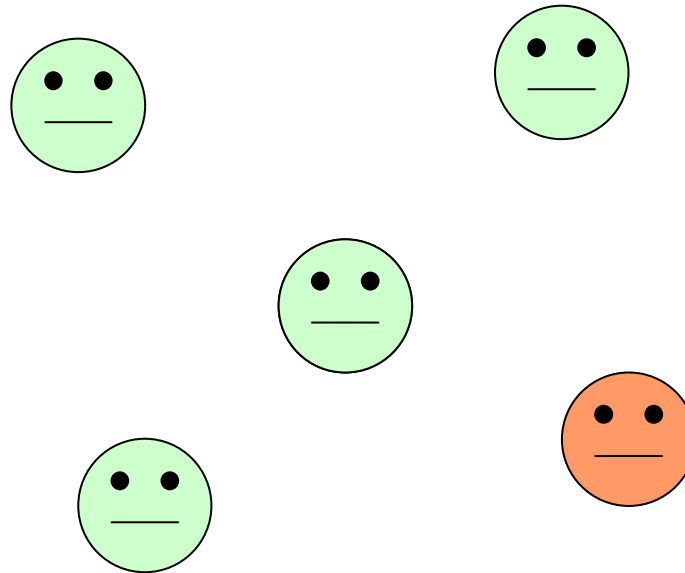
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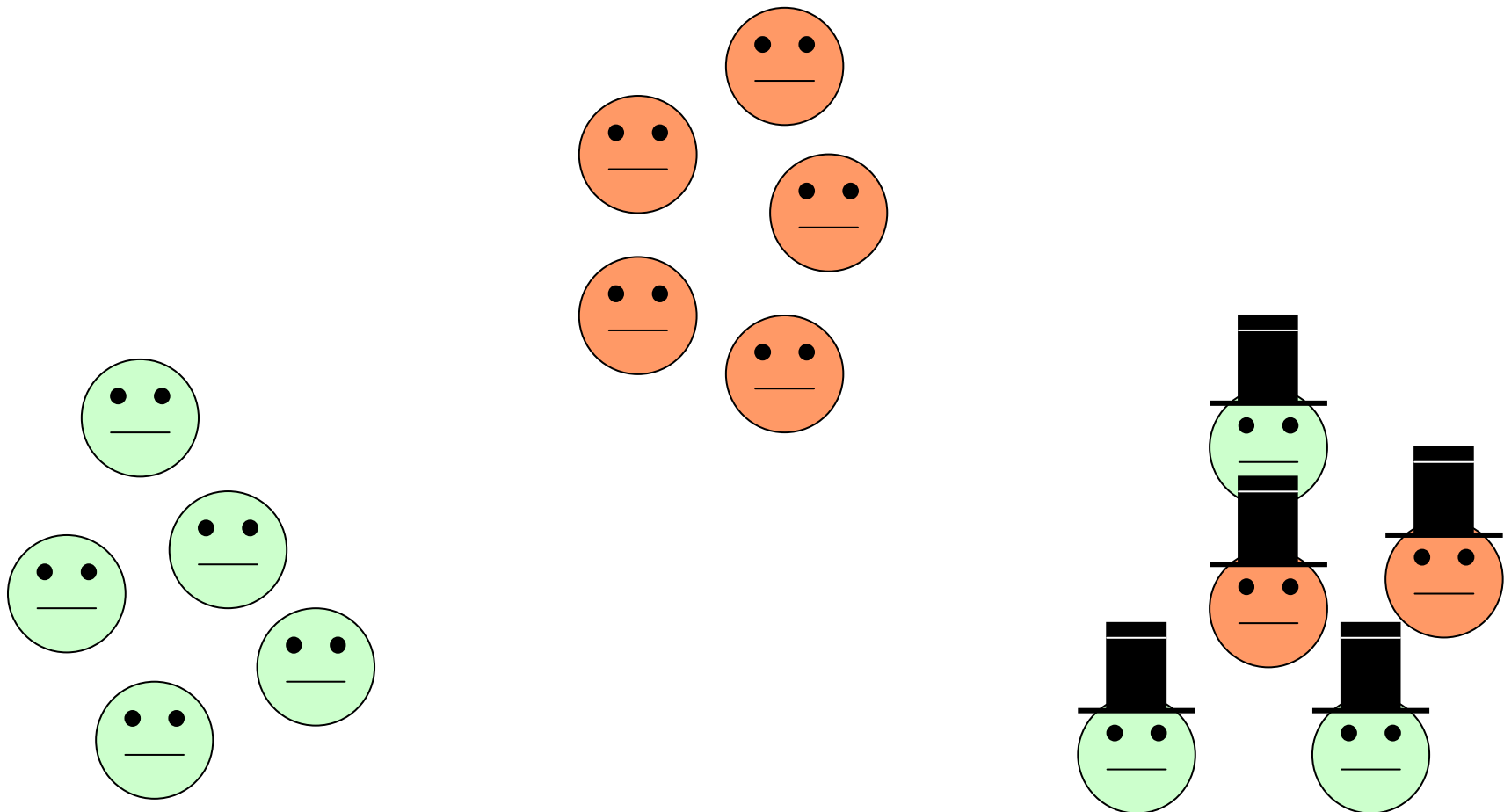
K. Suchecki, J. A. Hołyst , PHYS REV E **74**; 011122 Part 1 JUL 2006

K. Suchecki, J.A. Hołyst, arxiv 0802.1499

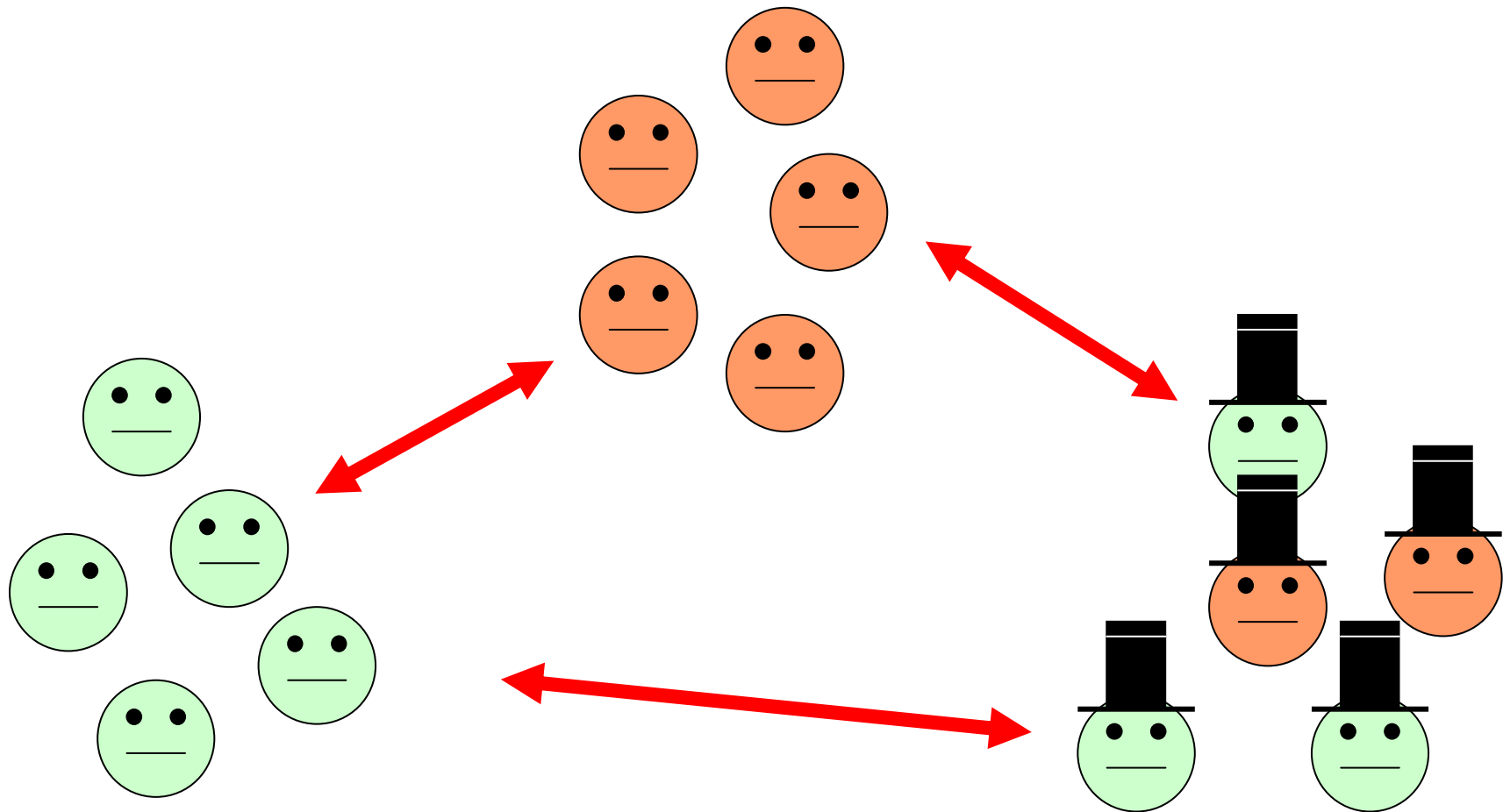
- Social interactions – often majority-driven



- Social interaction networks are often modular (subgroups)



- There is often a **competition** between different subgroups



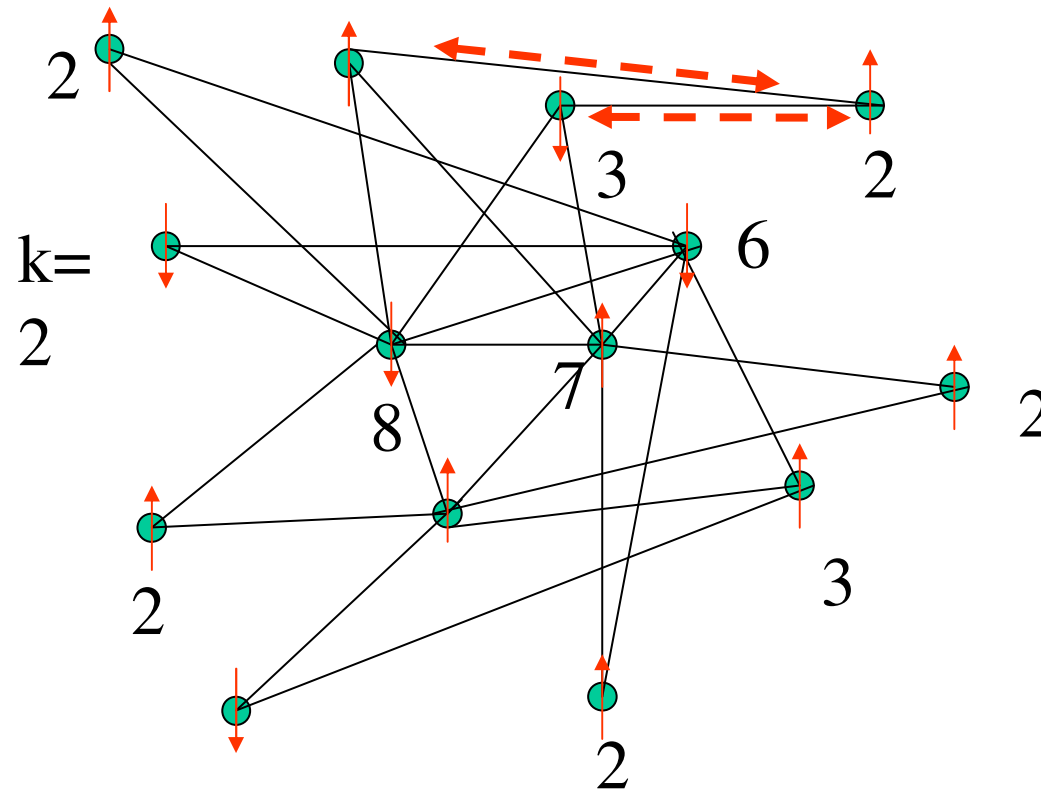
What is a fitness function for social groups competition ?

- Size ?
- Density of internal connections ?
- Strength of internal links ?
- Distribution of links ?

How to measure it ?

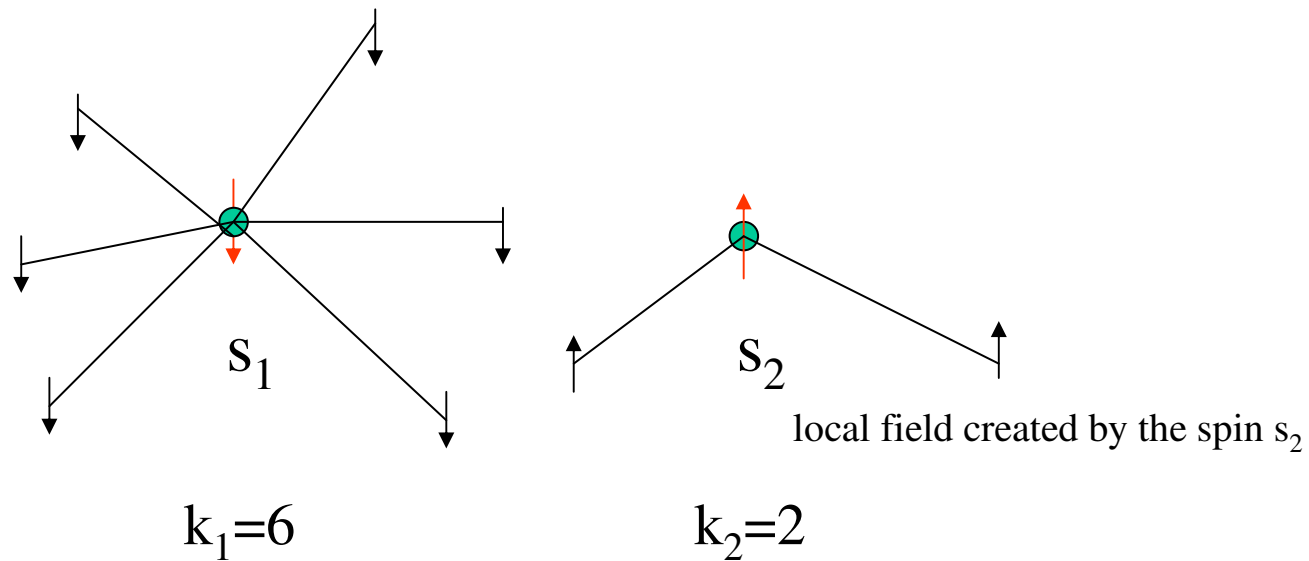
Introduction

- We represent simplest social interactions with Ising model on network



Scale-free networks

What does the weighted spin mean, and why weighted ?



$$s_1 + s_2 = 0 \quad \text{no order ?}$$

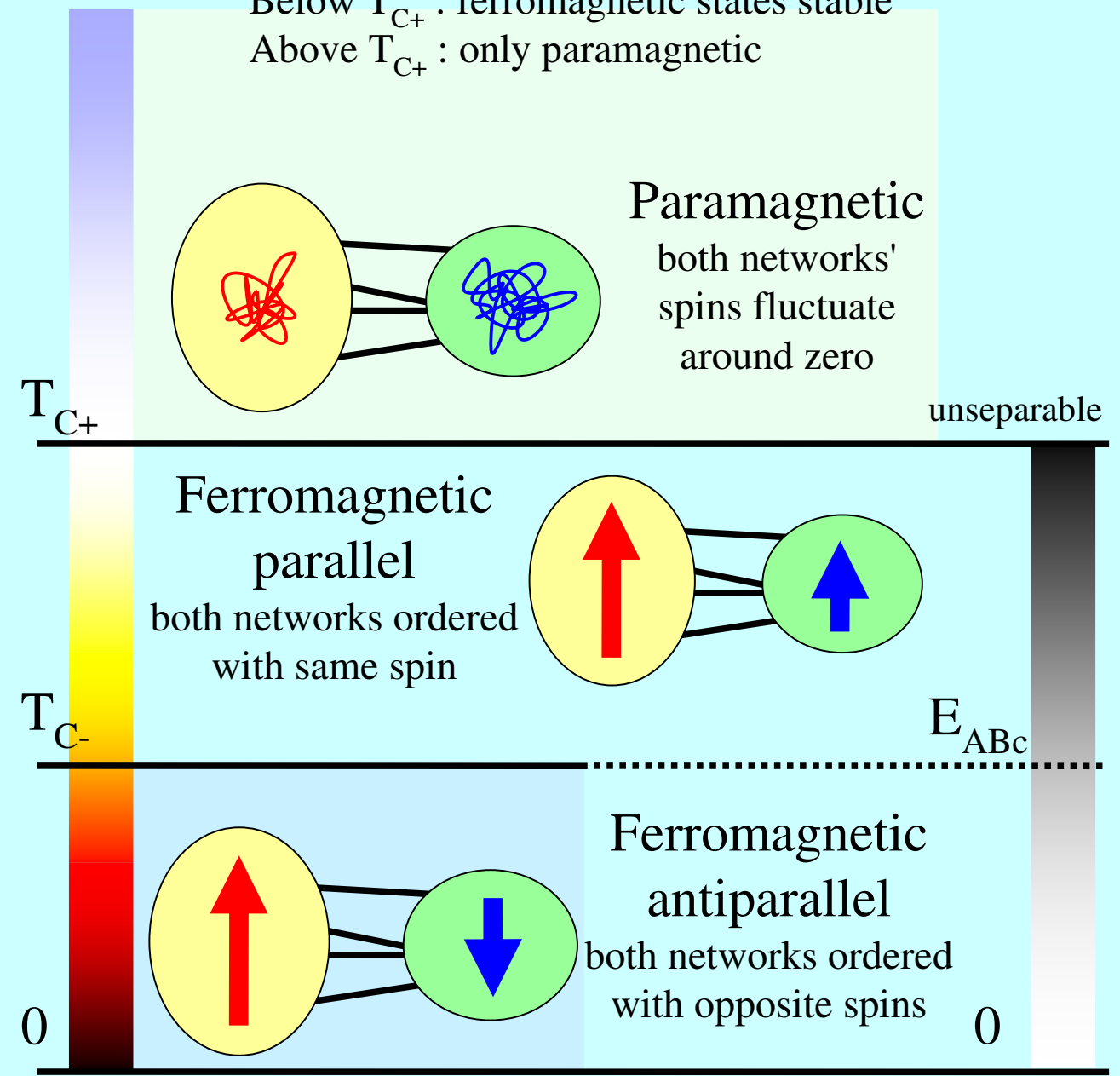
$$s_1 k_1 + s_2 k_2 < 0 \quad \text{order !}$$

Temperature and internetwork interaction strength have similar effect on antiparallel-parallel transition.

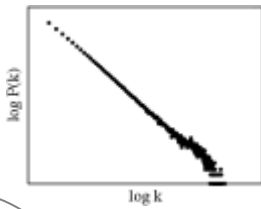
Networks become unseparable at internetwork density equal to intranetwork density.

STABLE STATES

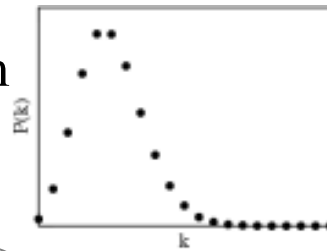
Below T_{C+} : ferromagnetic states stable
 Above T_{C+} : only paramagnetic



scale-free
B-A



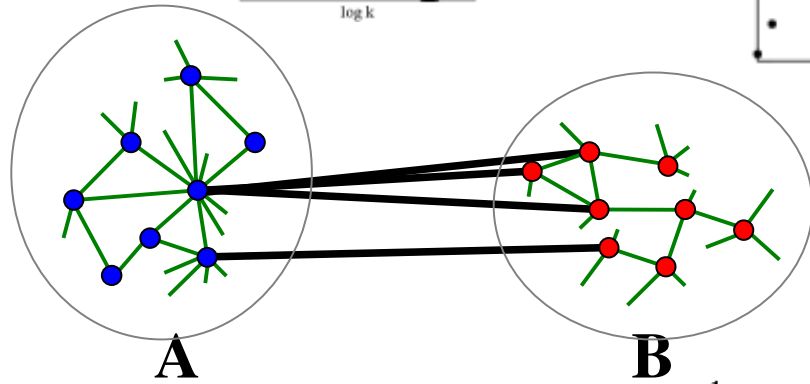
random
E-R



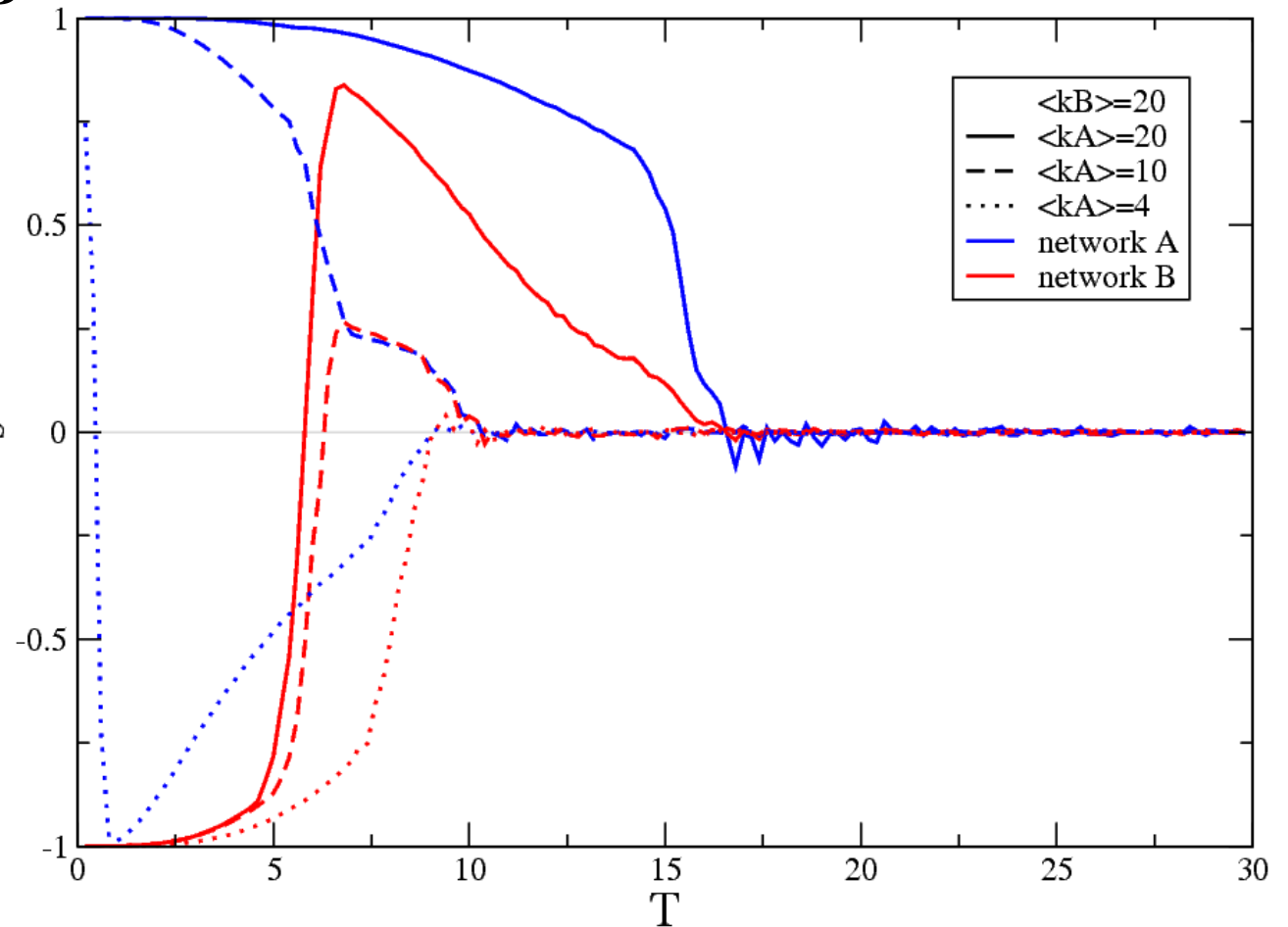
Net A : B-A, Net B : E-R

$N_A = N_B = 100$, $E = 100$

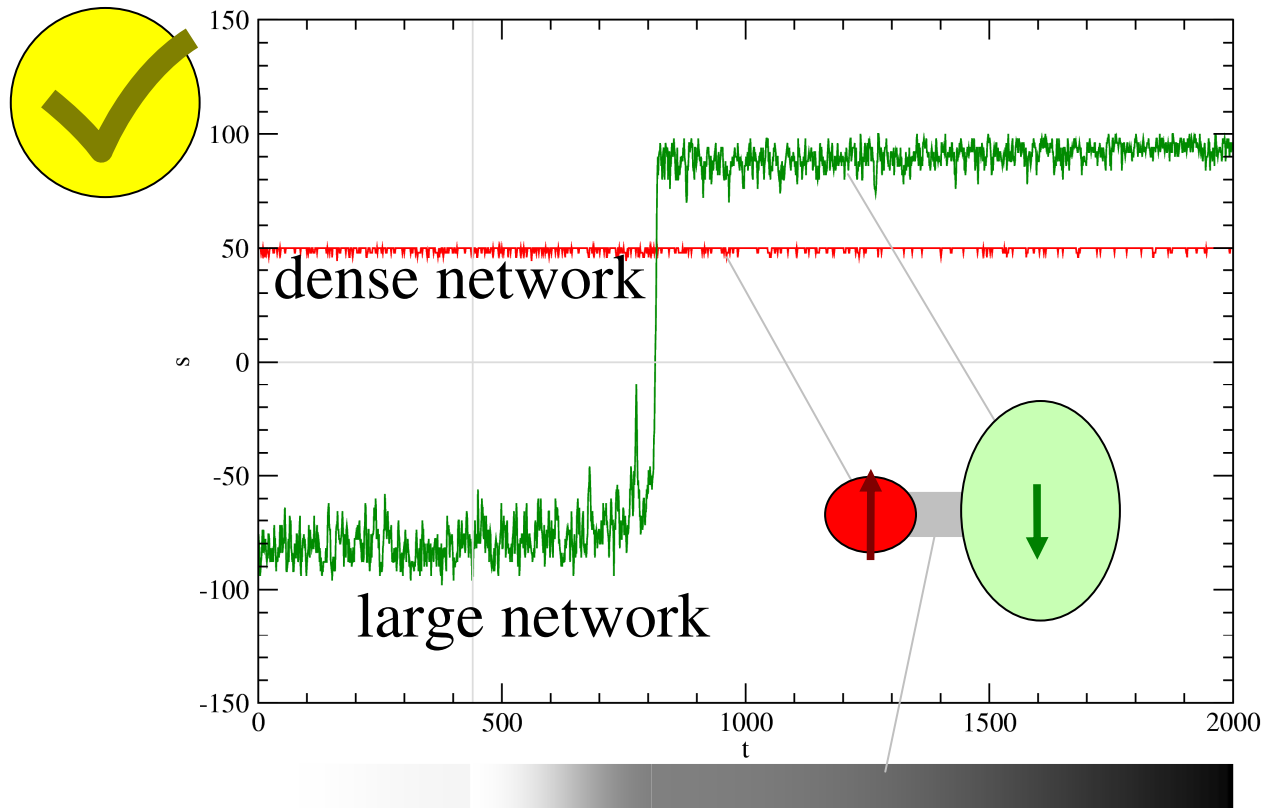
$\langle k_B \rangle = 20$, various $\langle k_A \rangle$



Scale free B-A network
wins a competition against
a random E-R graph



fluctuations



Network A (red) is smaller, but denser.
 Network B (green) is larger, but sparse.

The internetwork connections are added with time

The network with larger fluctuation loses

Details	
	$\langle k \rangle$
A	8
B	4

Network with small “strength” suffers large fluctuations – fluctuating networks flip over first and therefore lose

Susceptibility –a measure how external field changes order parameter

Susceptibility $\chi = \frac{\partial S_A}{\partial \lambda}$

order parameter
interaction λ

Fluctuation-dissipation theorem

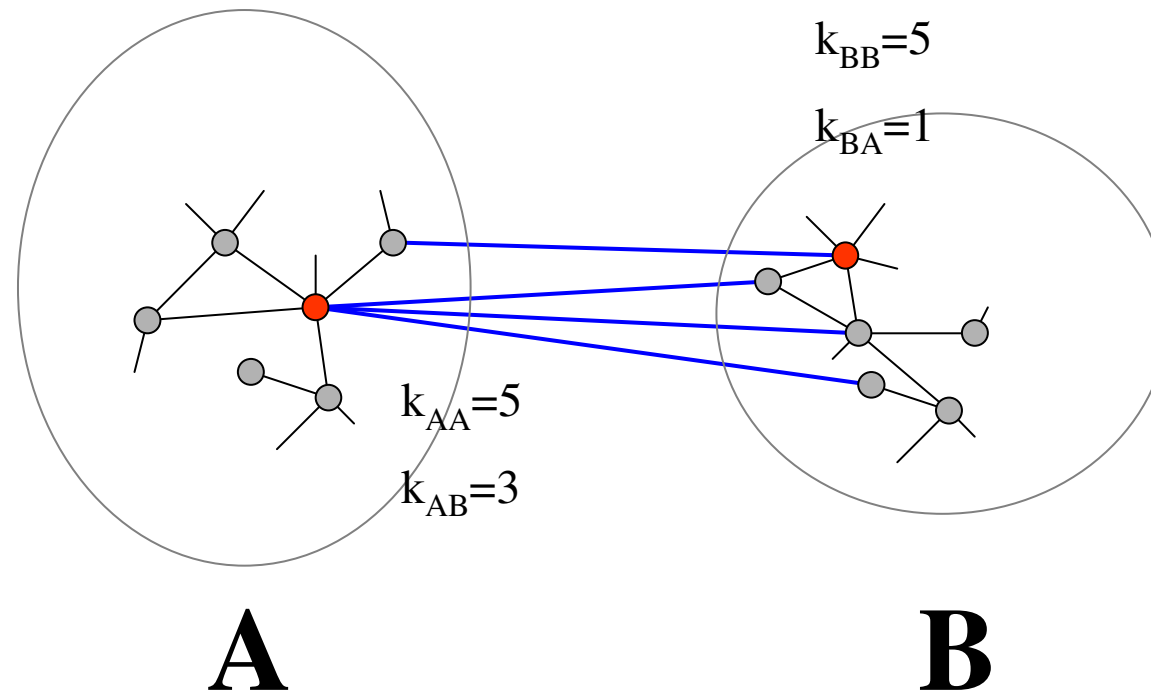
$$\chi = \frac{\partial S_A}{\partial \lambda} = \text{susceptibility} \sim \text{fluctuations} = \langle (S_A - \langle S_A \rangle)^2 \rangle$$

small fluctuations \rightarrow WIN

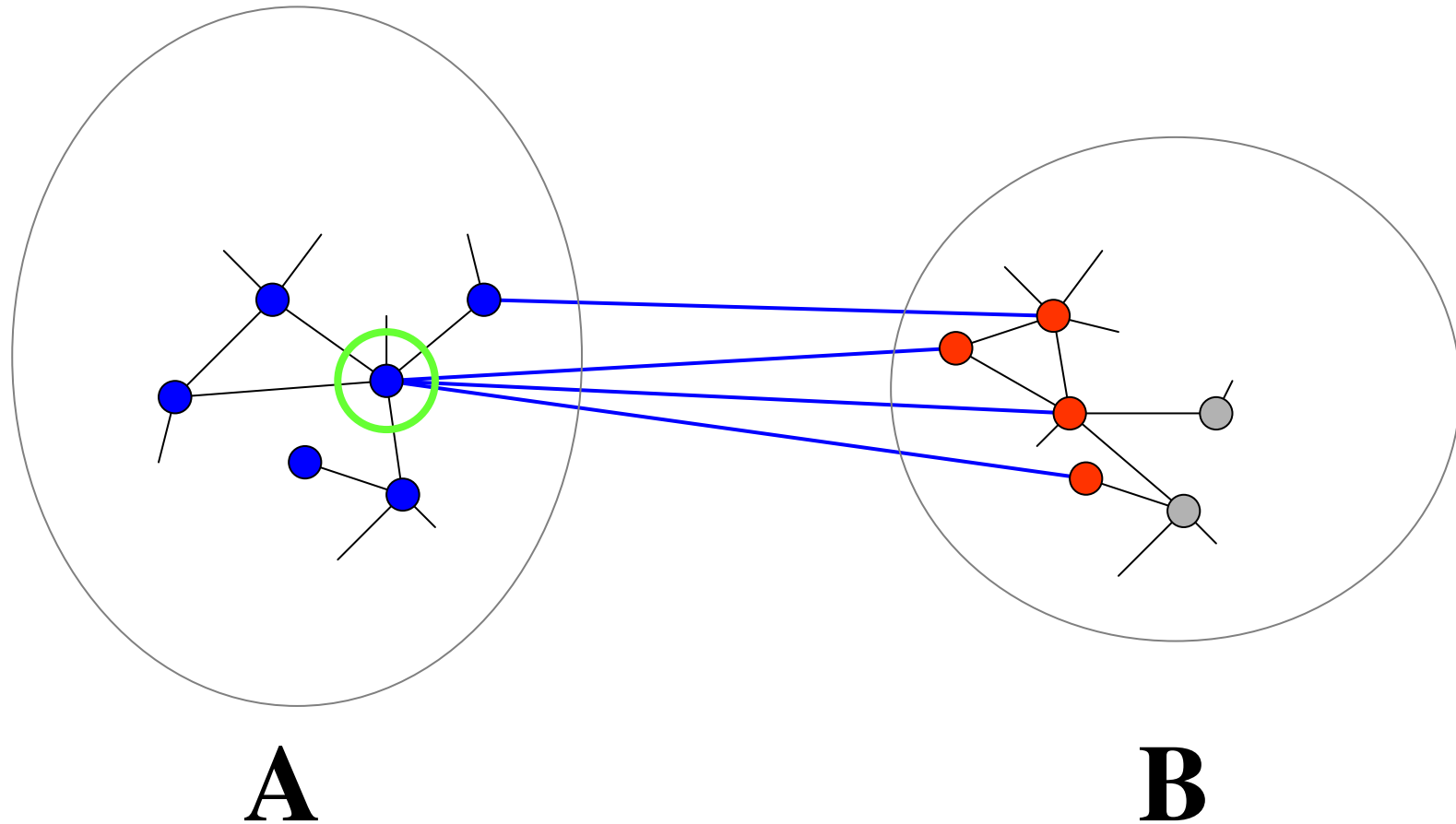
because smaller fluctuations \rightarrow smaller susceptibility thus other network (with larger susceptibility) changes spin faster

Theory of coupled networks

System of two coupled networks (one larger modular network)



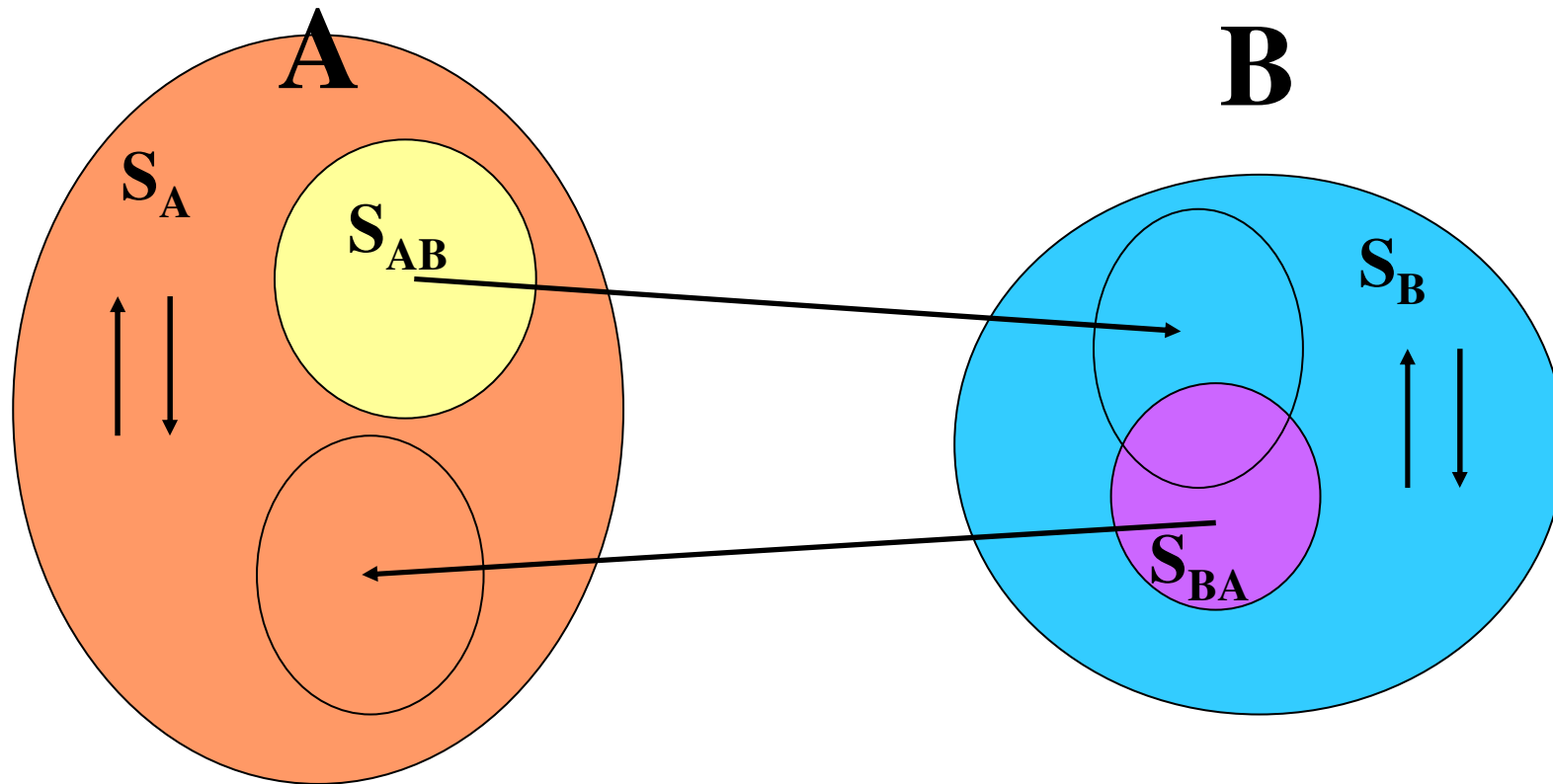
Coupled networks



S_A - spin of network A weighted by k_{AA} ()

S_{BA} - spin of network B weighted by k_{BA} ()

Coupled networks



The state of the system can be written as vector:

$$\begin{pmatrix} S_A \\ S_{AB} \\ S_{BA} \\ S_B \end{pmatrix}$$

Coupled networks

If we assume that the number of inter-network connections is proportional to the intra-network degree:

$$k_{AB} = p_A k_{AA}, \quad k_{BA} = p_B k_{BB}$$

then we do not have to consider S_{AB} and S_{BA} anymore since they are proportional to S_A and S_B .

$$\begin{bmatrix} \Lambda_{AA} & \Lambda_{BA} \\ \Lambda_{AB} & \Lambda_{BB} \end{bmatrix} \begin{pmatrix} S_A \\ S_B \end{pmatrix} = \lambda \begin{pmatrix} S_A \\ S_B \end{pmatrix}$$

Coupled networks

We have following eigenvalues λ of the matrix Λ :

$$\lambda_{\pm} = \frac{\Lambda_{AA} + \Lambda_{BB} \pm \sqrt{(\Lambda_{AA} - \Lambda_{BB})^2 + 4\Lambda_{BA}\Lambda_{AB}}}{2}$$

λ_- : eigenvector $\begin{pmatrix} 1 \\ -c_1 \end{pmatrix}$

The networks are ordered antiparalelly. T_C is lower than for separate networks.

Increasing inter-network connection strengths causes T_C to decrease, down to 0, when the inter-network connections are as dense as intra-network.

λ_+ : eigenvector $\begin{pmatrix} 1 \\ c_2 \end{pmatrix}$

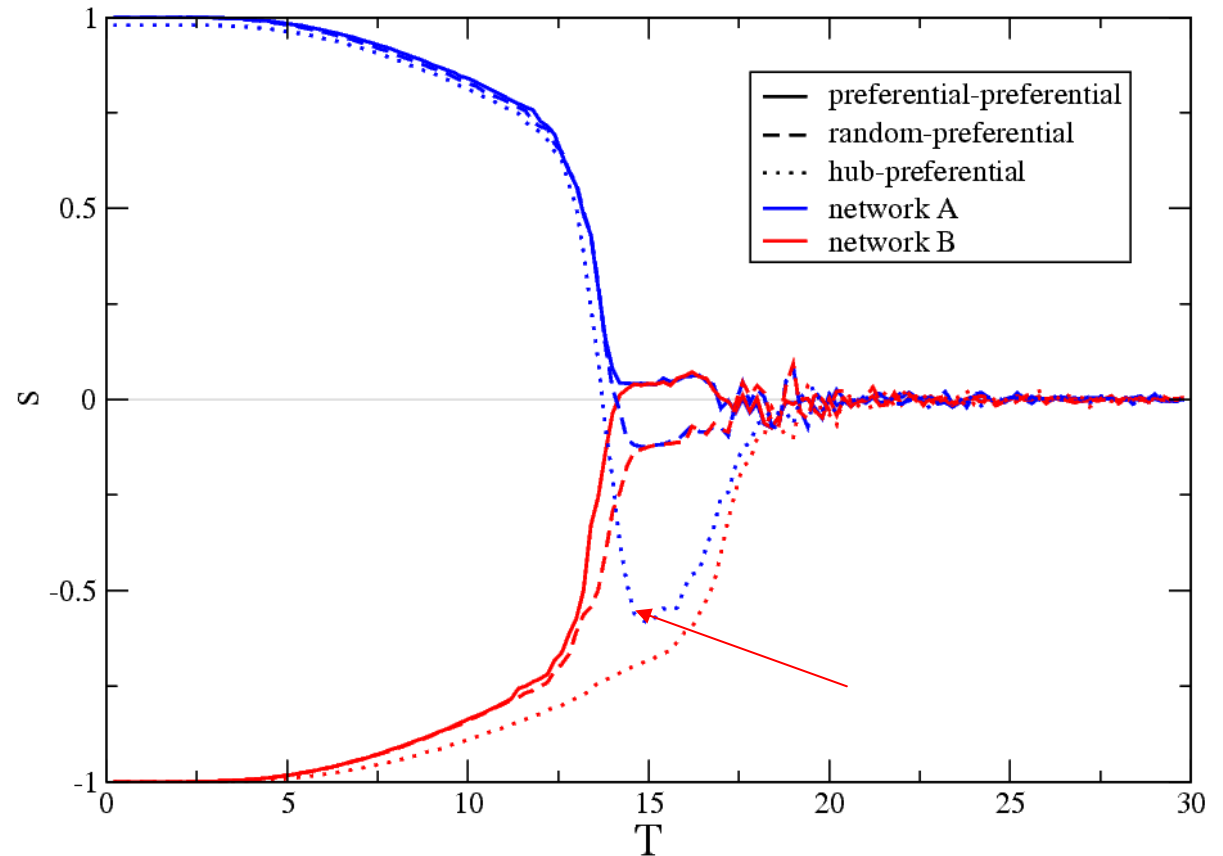
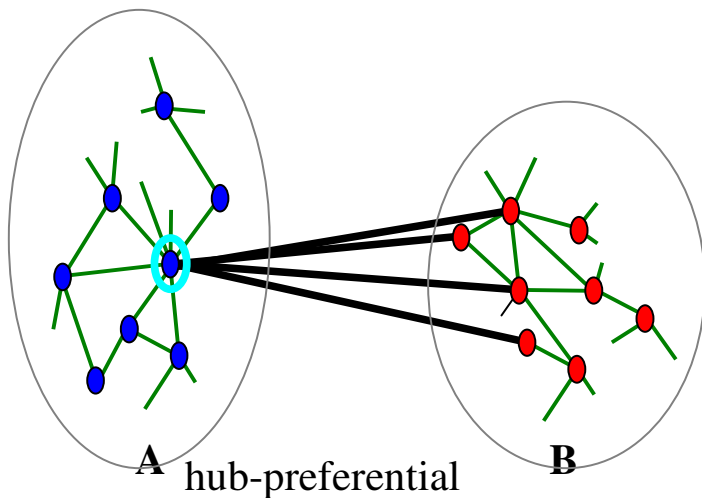
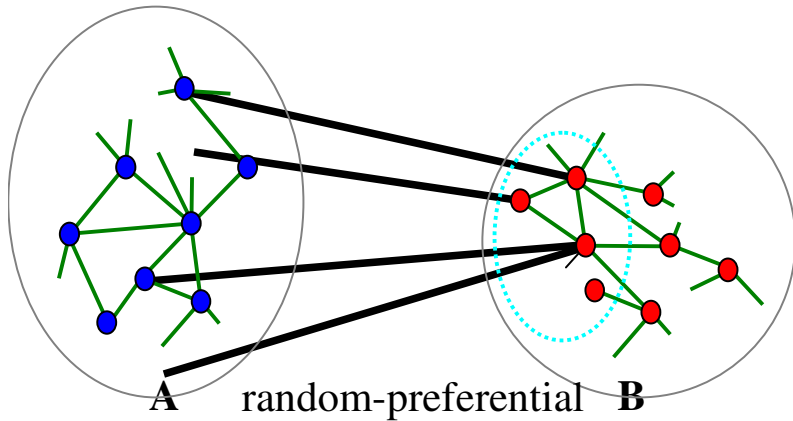
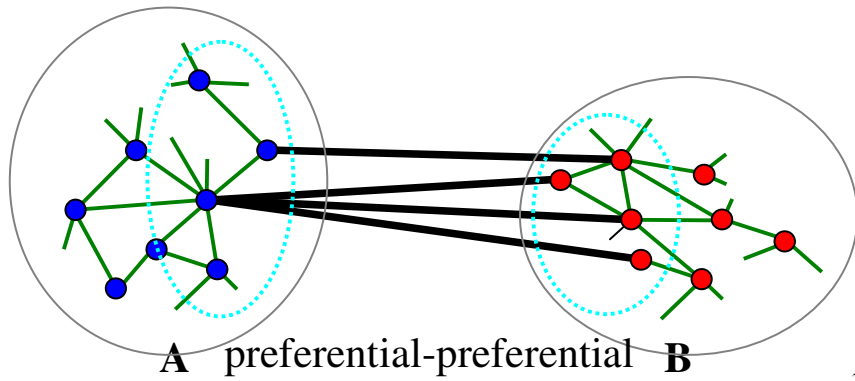
The networks are ordered paralelly. T_C is higher than for separate networks.

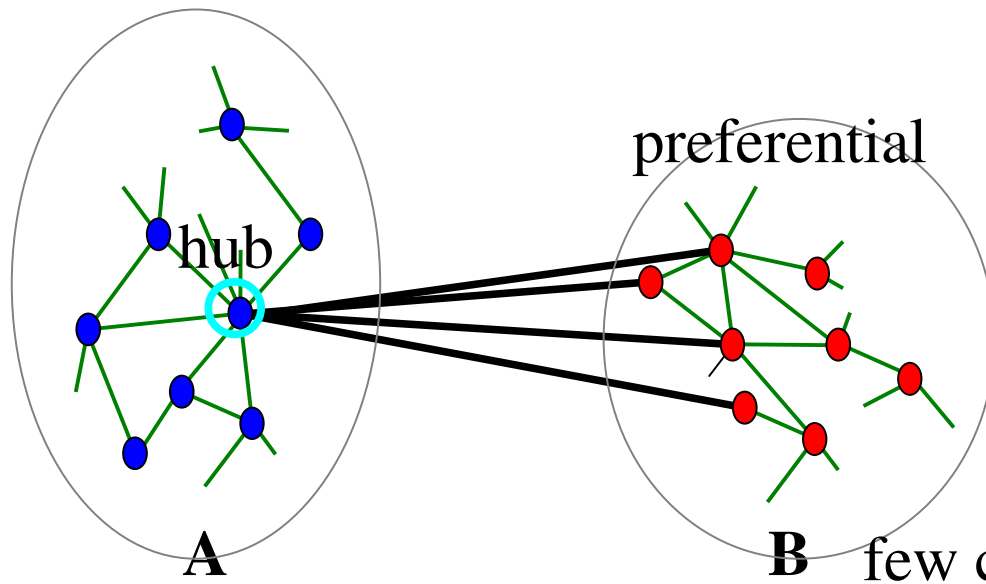
Networks stabilize each other, similar to way the network stabilizes itself.

$$N_A = N_B = 100, \langle k_A \rangle = \langle k_B \rangle = 20$$

$E = 100$ (number of inter-network connections)

There are different scenarios by the same networks but different internetwork connections





$$N_A = N_B = 100$$

$$\langle k_A \rangle = 20$$

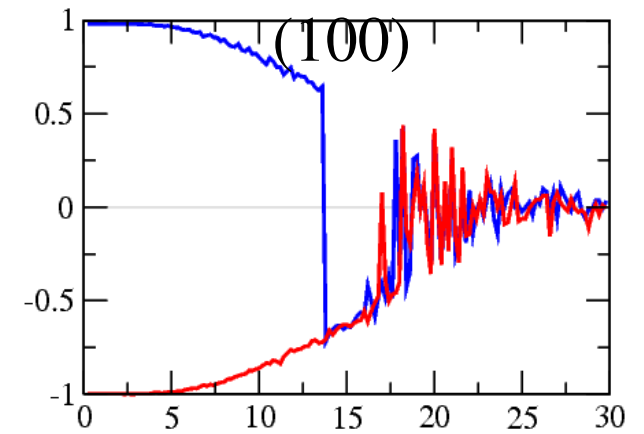
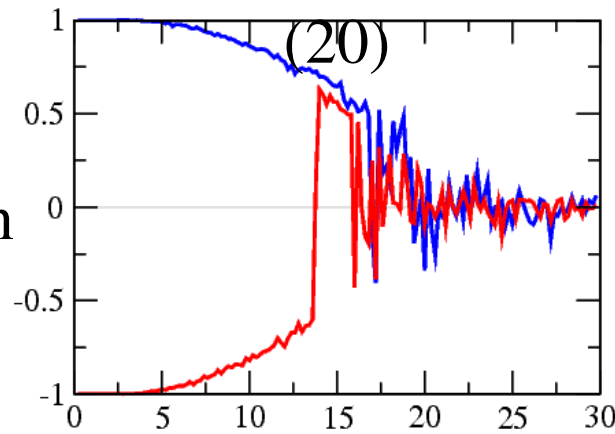
$$\langle k_B \rangle = 18 \quad \text{network B is "weaker"}$$

$s(T)$

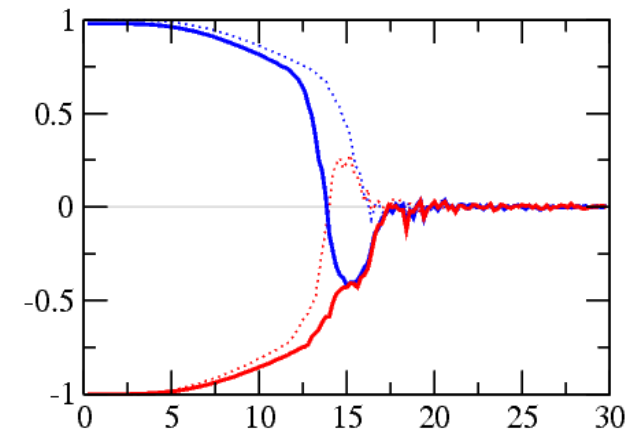
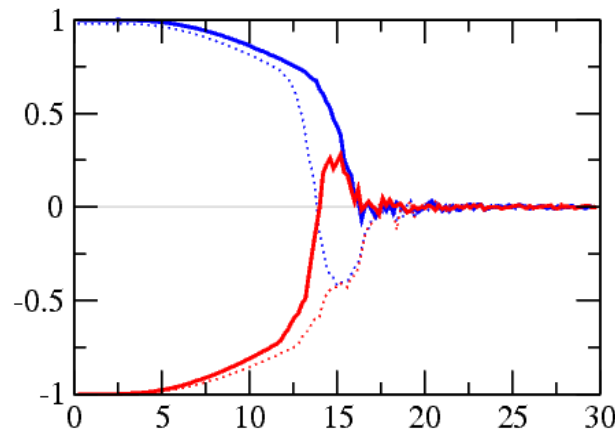
B few connections

many connections

single
simulation



averaged
(100)

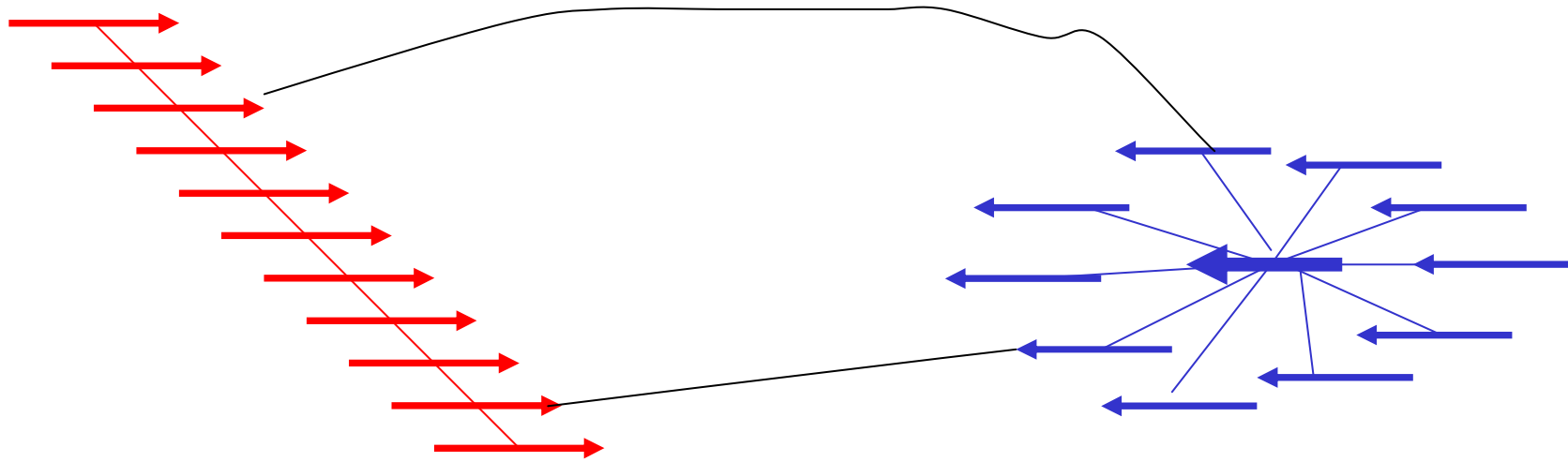


**Nec Hercules
contra plures !**

Conclusions

- Competition of two social groups can bring a discontinuous change in their opinions as a function of intergroup connections or noise level
- Larger group is not **always** a winner !
- Density and **distribution** of internal links is also important
- Winner can depend on the **choice** of group-group interactions
- Observation of proper **fluctuations** before the competition can be a useful indicator.

The simplest confrontation: **chain** and star

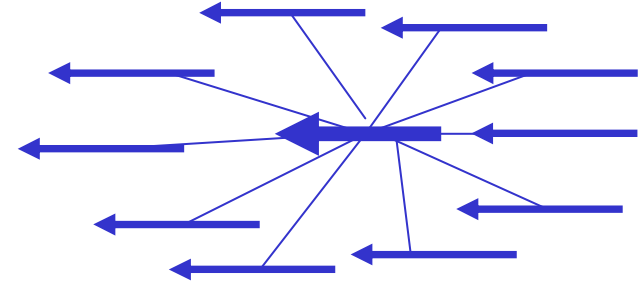


$$H_{chain} = -J \sum_{i=1}^N S_i S_{i+1}$$

$$H_{star} = -JS_0 \sum_{i=1}^N S_i$$

Statistical physics of a star network

$$E_k^{star} = -J(+1)k - J(-1)(N - k)$$

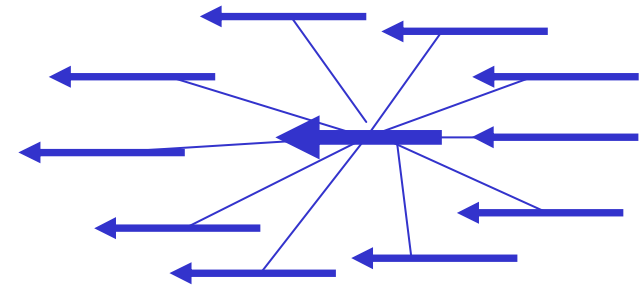


$$Z_{star} = \sum_{s_0, \{s_i\} = -1, 1} \exp(-\beta E_k) = 2 \sum_{k=0}^N \binom{N}{k} \exp(\beta J k) \exp[-\beta J (N - k)]$$

$$Z_{star} = \dots = 2(e^{-\beta J} + e^{+\beta J})^N = Z_{chain} !!!$$

Star network in homogenous external magnetic field

$$H_{star} = -JS_0 \sum_{i=1}^N S_i - B \sum_{i=0}^N S_i$$



$$\langle s_0 s_i \rangle = \tanh(\beta J) \quad \langle s_j s_i \rangle = \tanh^2(\beta J)$$

$$\chi_{star} = \frac{\partial \langle S \rangle}{\partial B} = \sum_{i,j=0}^N \langle s_i s_j \rangle = \beta \left[(N+1) + 2N \tanh(\beta J) + N(N-1) \tanh^2(\beta J) \right]$$

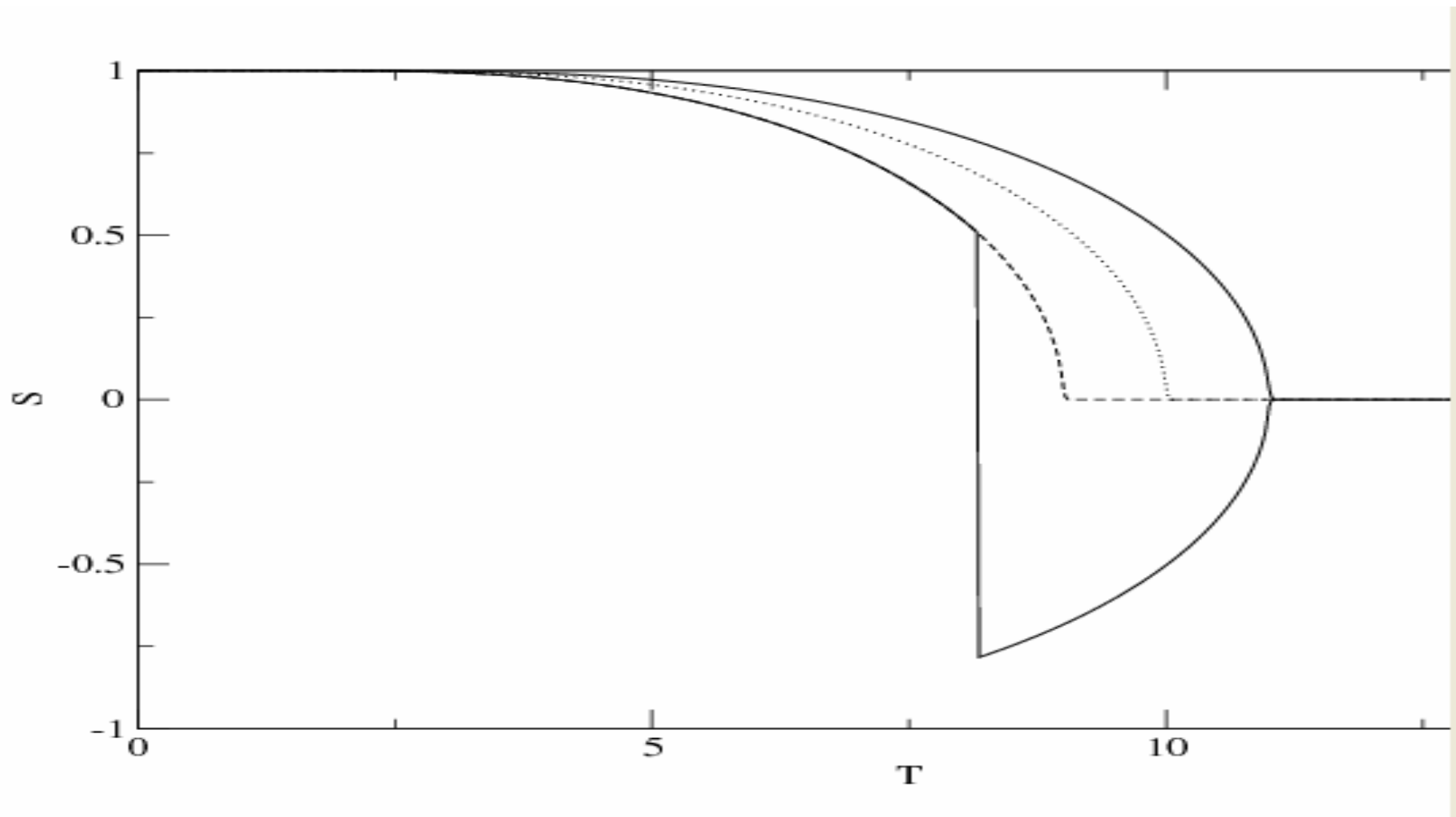
$$\chi_{star} / N \approx N \beta \tanh^2(\beta J)$$

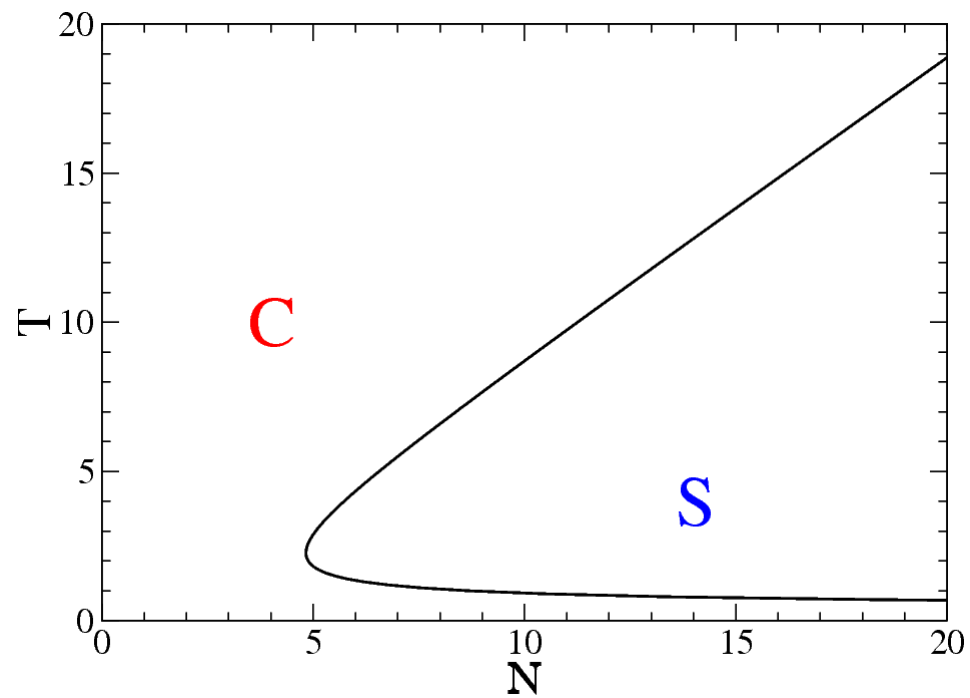
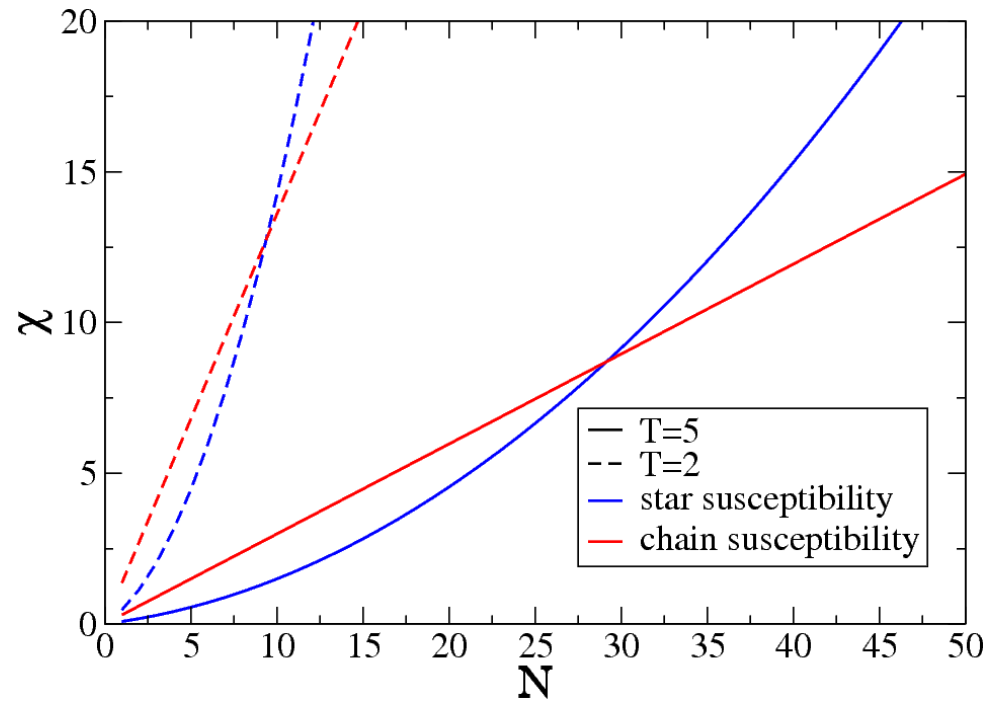
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Conclusions

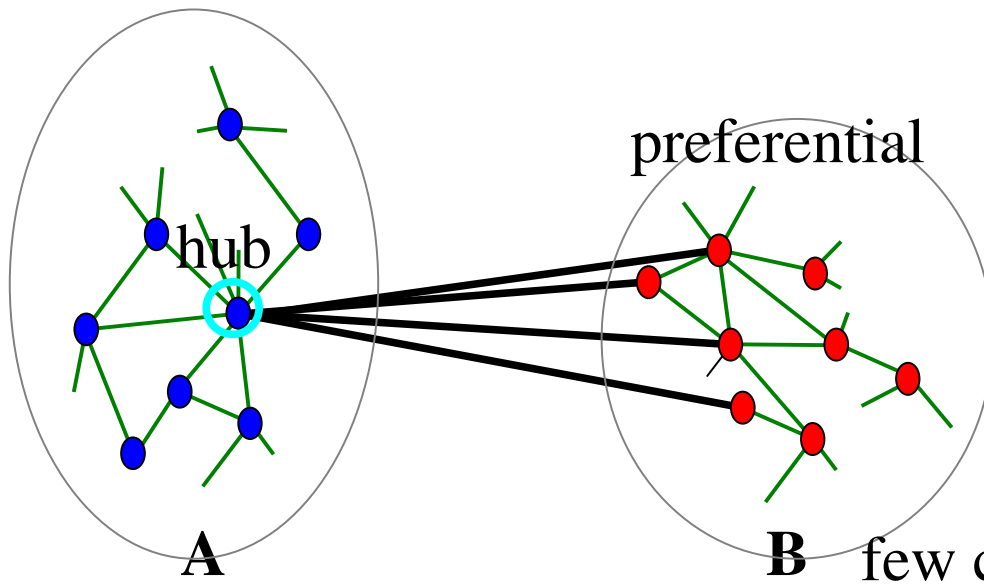
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First order phase transition in Ising model on two connected Barabasi-Albert networks





Curve of
susceptibility
balance between
spin chain and
spin star



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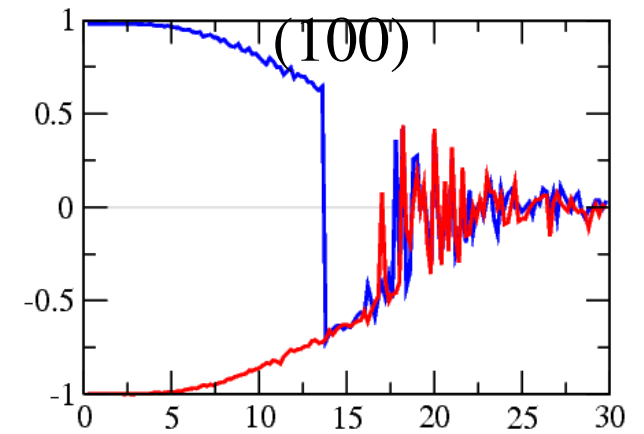
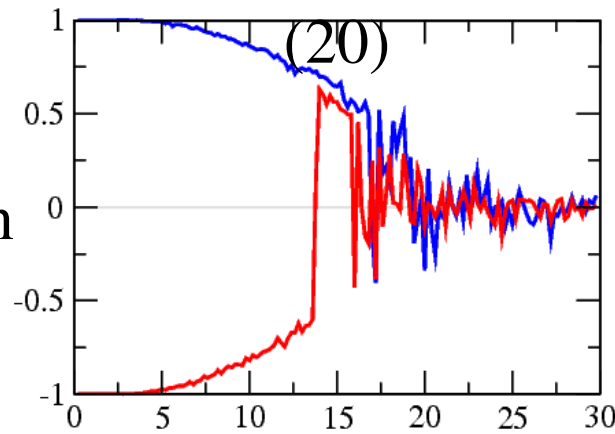
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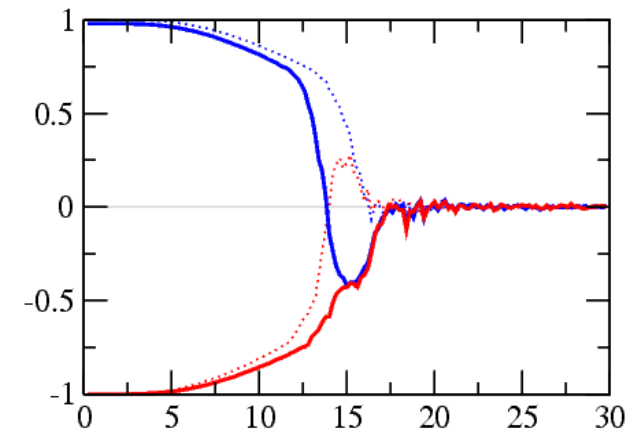
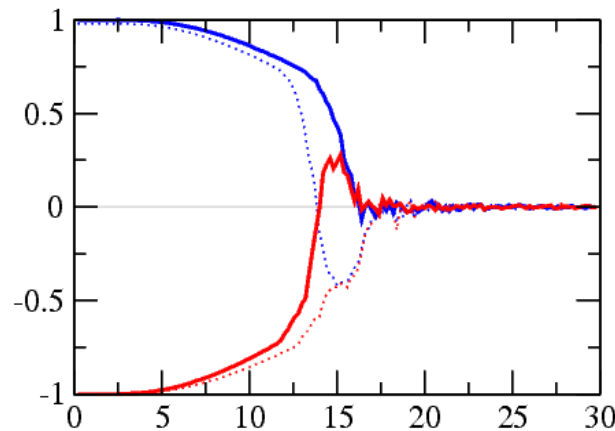
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many connections

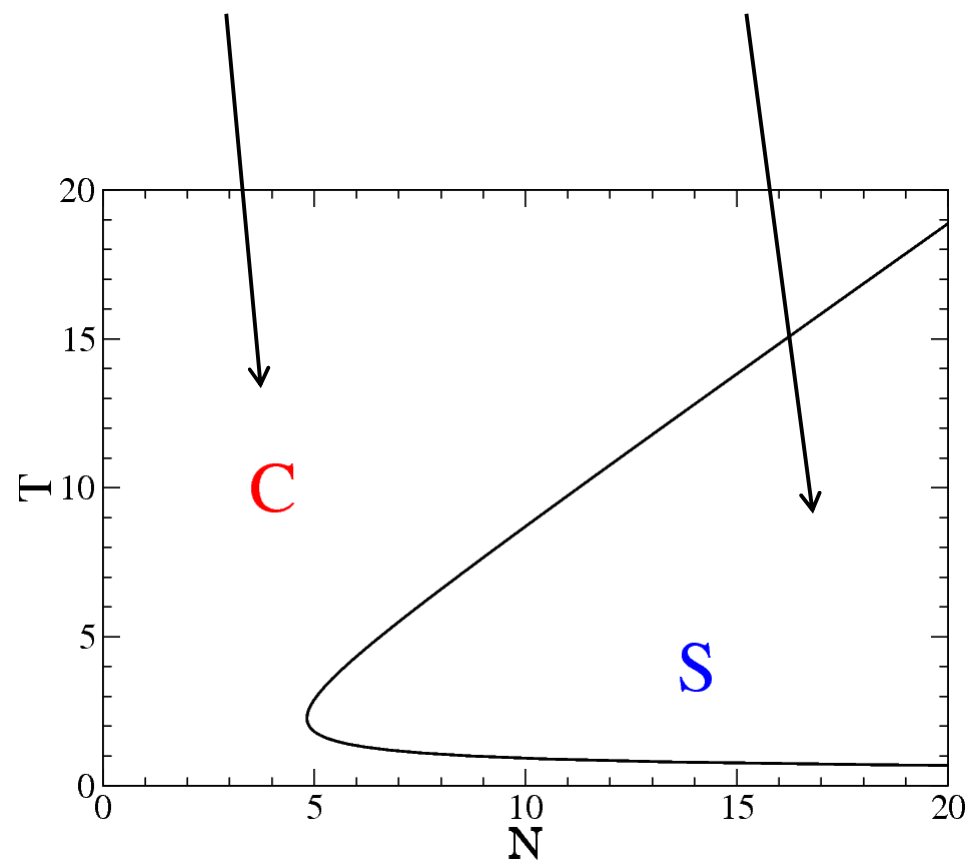
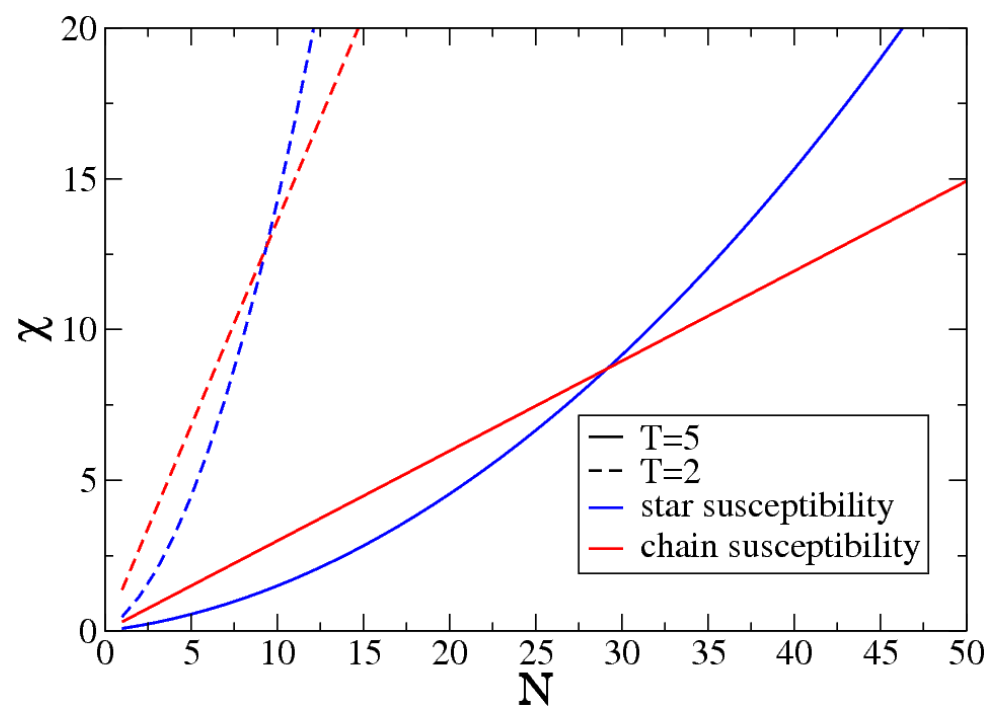
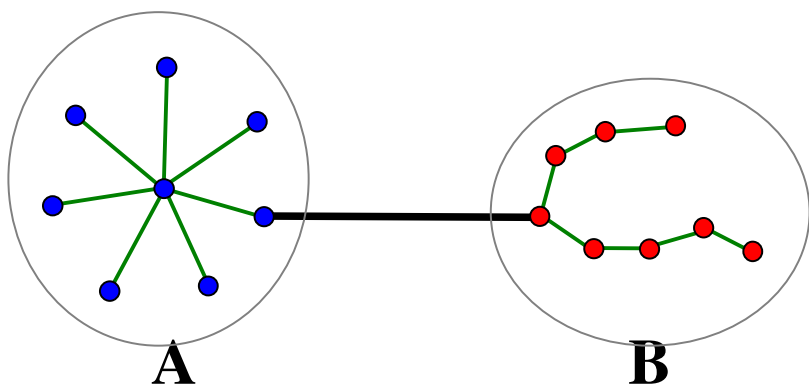
single
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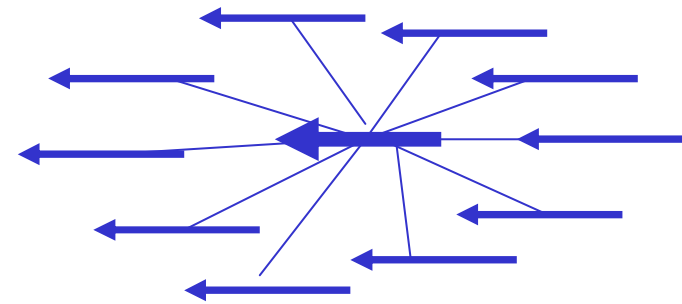
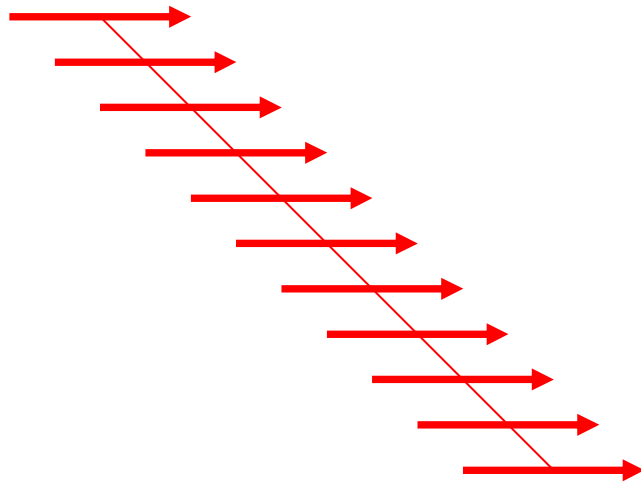
averaged
(100)



**Nec Hercules
contra plures !**



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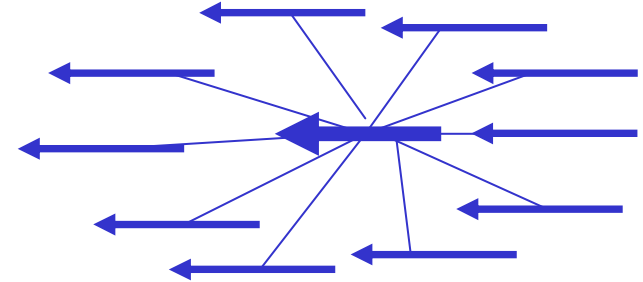


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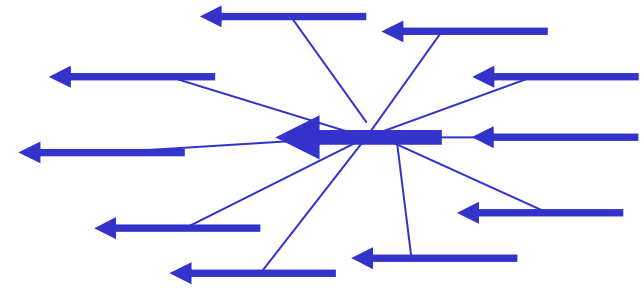


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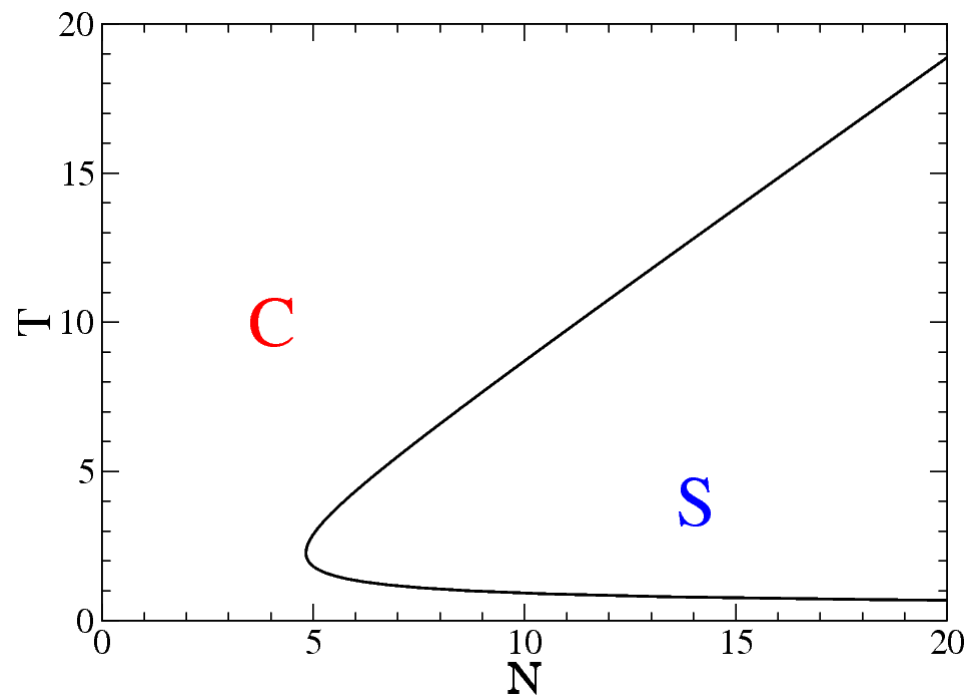
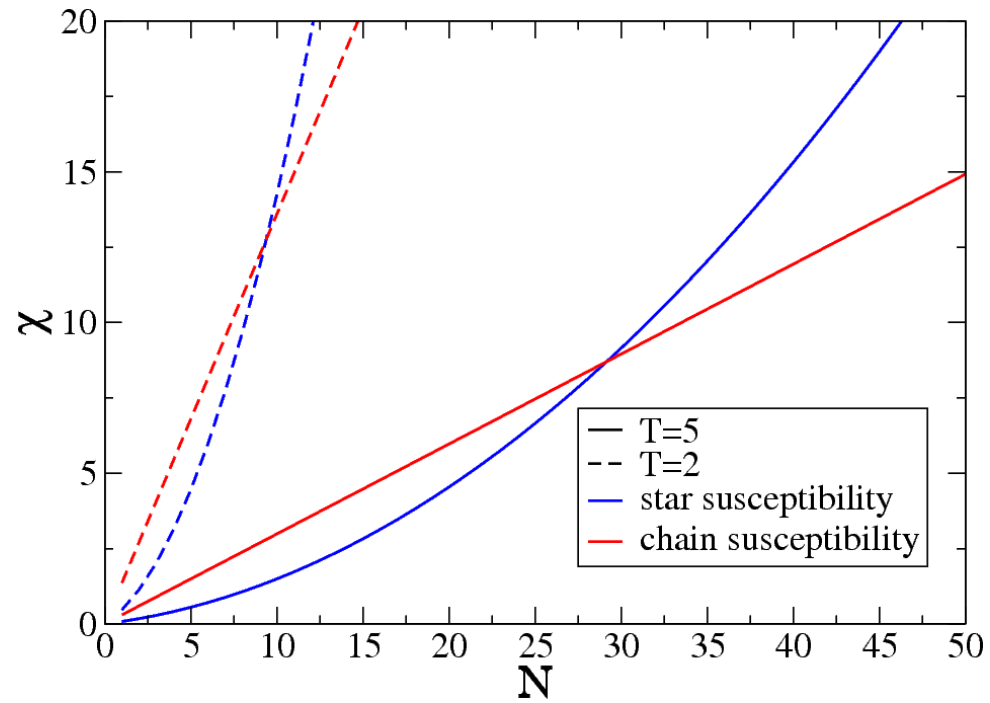


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$$\chi_{chain} / N \approx \beta \exp(2\beta J)$$



Curve of
susceptibility
balance between
spin chain and
spin star

Statistical Physics of Hamiltonian Graphs

$$S = - \sum_G P(G) \ln P(G), \quad (1)$$

subject to the constraints

$$\langle x_i \rangle = \sum_G x_i(G) P(G) \quad (2)$$

for $i=1, 2, \dots, r$, plus the normalization condition

$$\sum_G P(G) = 1. \quad (3)$$

The Lagrangian for the above problem is given by the expression

$$\mathcal{L} = - \sum_G P(G) \ln P(G) + \alpha \left(1 - \sum_G P(G) \right) \quad (4)$$

$$+ \sum_{i=1}^r \theta_i \left(\langle x_i \rangle - \sum_G x_i(G) P(G) \right), \quad (5)$$

$$P(G) = \frac{e^{-H(G)}}{Z}, \quad (6)$$

where $H(G)$ is the network Hamiltonian

$$H(G) = \sum_{i=1}^r \theta_i x_i(G), \quad (7)$$

and Z represents the partition function (normalization constant)

$$Z = \sum_G e^{-H(G)} = e^{\alpha+1}. \quad (8)$$

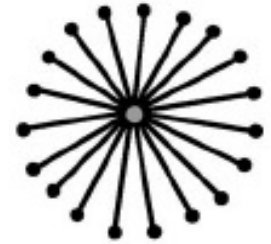
Fluctuation-dissipation relations for Hamiltonian graphs

$$\langle x_i^2 \rangle - \langle x_i \rangle^2 = - \frac{\partial \langle x_i \rangle}{\partial \theta_i} = \chi_i^{(\theta)}$$

$$\langle E^2 \rangle - \langle E \rangle^2 = - \frac{\partial E}{\partial \beta} = kT^2 C_V$$

$$\chi_i^{(\theta)} = - \frac{\partial \langle k_i \rangle}{\partial \theta_i} = \langle k_i^2 \rangle - \langle k_i \rangle^2 = \sum_j p_{ij}(1 - p_{ij}) = \langle k_i \rangle - \sum_j p_{ij}^2$$

Fluctuation-dissipation relations for Hamiltonian stars



$$\langle k^* \rangle = N$$

$$\chi = - \frac{1}{\langle k \rangle} \frac{\partial \langle k \rangle}{\partial \theta} = 1 - \frac{(N - \langle k^* \rangle)^2}{N^2(N - 1)} - \frac{\langle k^* \rangle^2}{N^2}$$

for the bulk of nodes and

$$\chi^* = - \frac{1}{\langle k^* \rangle} \frac{\partial \langle k^* \rangle}{\partial \theta^*} = 1 - \frac{\langle k^* \rangle}{N}$$

for the supernode.

- K. Suchecki, Janusz A. Hołyst , *Ising model on two connected Barabasi-Albert networks*, PHYS REV E **74**; 011122 Part 1 JUL 2006
- A. Fronczak, P. Fronczak, Janusz A. Hołyst , *Fluctuation-dissipation relations in complex networks*, Phys. Rev. E **73** 016108, JAN 2006

F A E N S

Wrocław
22-24.XI.2007

3 Ogólnopolskie Sympozjum Fizyka w Ekonomii i Naukach Społecznych



welcome

Zapraszamy do uczestnictwa w 3 Ogólnopolskim Sympozjum FENS 2007 poświęconym zastosowaniom metod i metodologii fizyki w szeroko rozumianej ekonomii, inżynierii finansowej, a także w naukach społecznych.

Nasze Sympozjum jest kontynuacją poprzednich konferencji w **Warszawie (2004)** i **Krakowie (2006)**, których materiały opublikowane zostały w **Acta Physica Polonica B** odpowiednio w **sierpniowym 2004** (1 Sympozjum) i **listopadowym 2006** (2 Sympozjum) numerze.

Celem Sympozjum jest:

1. Spotkanie fizyków, których tematyka badawcza obejmuje również obszary nauk ekonomicznych i społecznych.
2. Spotkanie fizyków z matematykami, ekonomistami, socjologami zainteresowanymi współpracą naukową oraz z praktykami finansów i biznesu (przedstawicielami banków, instytucji ubezpieczeniowych, firm komputerowych, giełdy, etc.).
3. Dyskusja nad problemami i potrzebami kształcenia studentów oraz doktorantów w zakresie ekono- i socjofizyki.

<http://www.science24.com/event/fens2007/>

Scaling and Universality in Proportional Elections

Santo Fortunato¹ and Claudio Castellano²

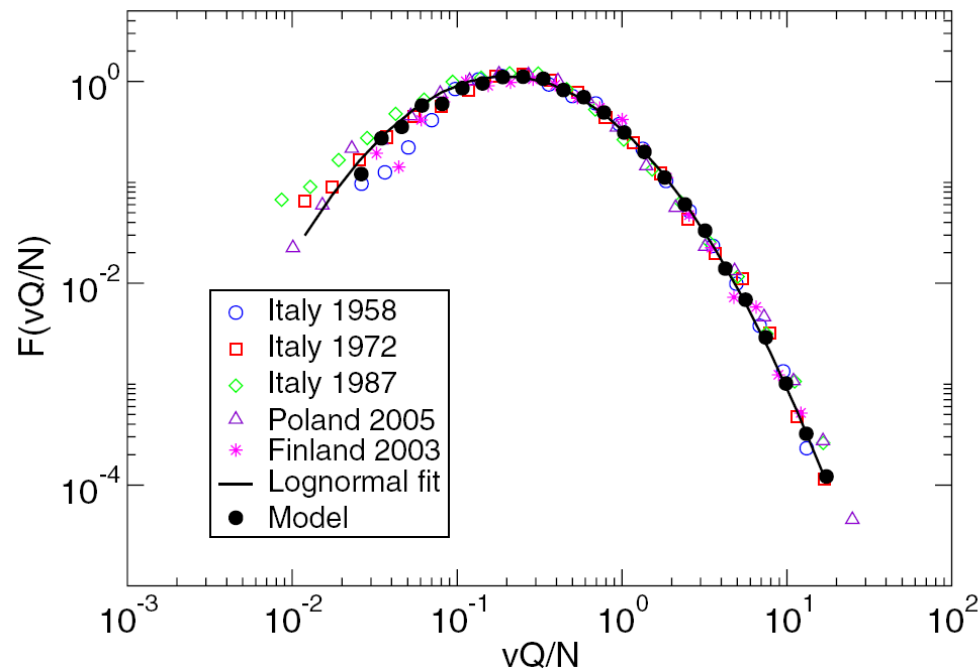


FIG. 2 (color online). Universality of the scaling function $F(vQ/N)$ across different countries and years. The lognormal fit, performed on the Polish curve, describes very well the data. The universal curve is well reproduced by our model, where the dynamics of the voters' opinions reflects the spreading of the word of mouth in the party's electorate.

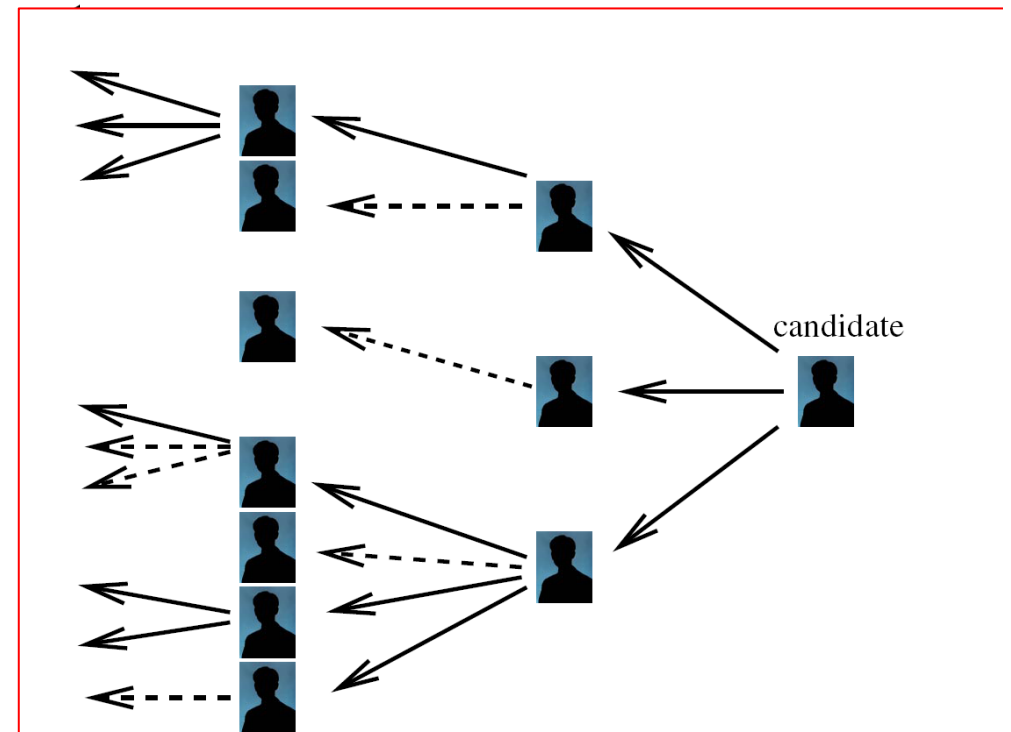


FIG. 3 (color online). Spreading of the word of mouth among voters. The candidate (right) convinces some of his or her contacts to vote for him/her. The convinced voters become "activists" and try to convince some of their acquaintances, and so on. Successful interactions are indicated by solid lines, unsuccessful interactions are displayed as dashed lines.

Regular networks

Hamiltonian for the Ising model, assuming the interaction constant is same for all pairs of spins.

$$H = -J \sum_{i,j} s_i s_j$$

Equation for mean spin (magnetization) in the mean-field approximation:

$$\langle s_i \rangle = \tanh \left(\beta \sum_j J_{ij} s_j \right) = \tanh(\beta J k \langle s_i \rangle)$$

k - coordination number (number of neighbors)

Scale-free networks

$$\langle s_i \rangle = \tanh \left(\beta \sum_j J_{ij} s_j \right) = \tanh \left(\beta J \sum_j \epsilon_{ij} \langle s_j \rangle \right)$$

Instead of coincidence matrix ϵ_{ij} we take the statistical probability of the spins being neighbors:

$$\langle \epsilon_{ij} \rangle = \frac{k_i k_j}{E} = \frac{k_i k_j}{\langle k \rangle N}$$

We obtain following

$$\langle s_i \rangle = \tanh \left(\frac{\beta J k_i}{\langle k \rangle N} \sum_j k_j \langle s_j \rangle \right)$$

Scale-free networks

$$\langle s_i \rangle = \tanh \left(\frac{\beta J k_i}{\langle k \rangle N} \sum_j k_j \langle s_j \rangle \right)$$

Now we define **average weighted spin** S as:

$$S = \frac{1}{\langle k \rangle N} \sum_i k_i \langle s_i \rangle$$

so we can write above as

$$S = \frac{1}{\langle k \rangle N} \sum_i k_i \tanh(\beta J k_i S)$$

Scale-free networks

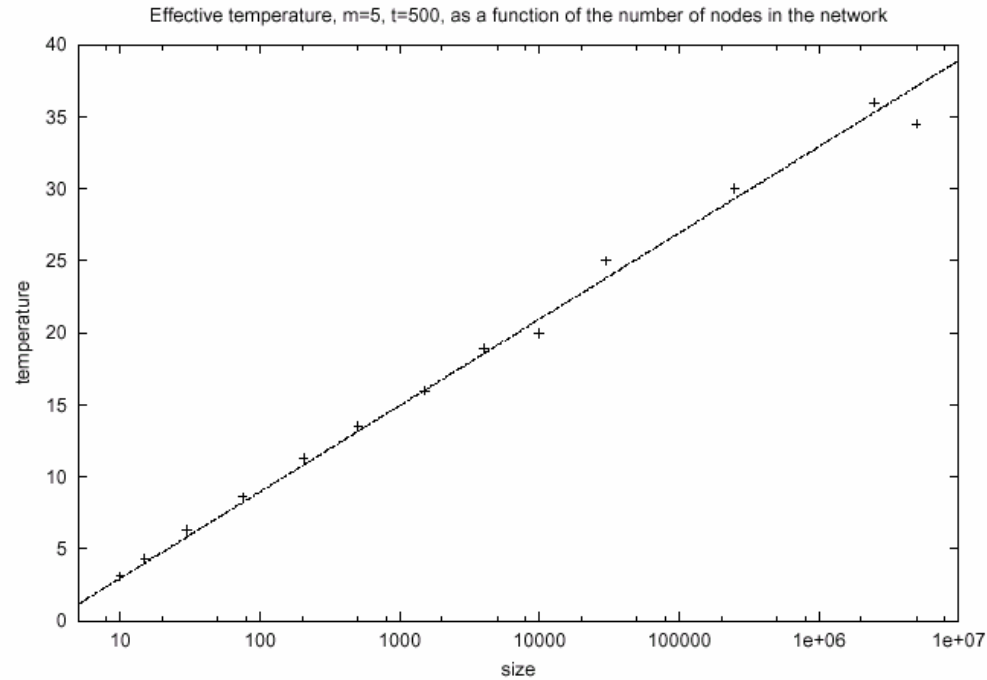
$$S = \frac{1}{\langle k \rangle_N} \sum_i k_i \tanh(\beta J k_i S)$$

Linear approximation

$$S = \frac{1}{\langle k \rangle_N} \sum_i k_i^2 \beta J S = \beta J \frac{\sum_i k_i^2}{\langle k \rangle_N} S = \beta J \frac{\langle k^2 \rangle}{\langle k \rangle} S$$

B-A network

$$\beta J \frac{\langle k^2 \rangle}{\langle k \rangle} S = \beta J \frac{m}{2} \ln NS \quad \longrightarrow \quad T_c = J \frac{m}{2} \ln N$$



Effective T_c versus $m+ N$ for $m = 5$ and various N , averaged over up to 1000 samples.



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Ferromagnetic phase transition in Barabási–Albert networks

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Coupled networks

Using linear approximation we investigate existence of nonzero solutions, that correspond to an ordered phase.

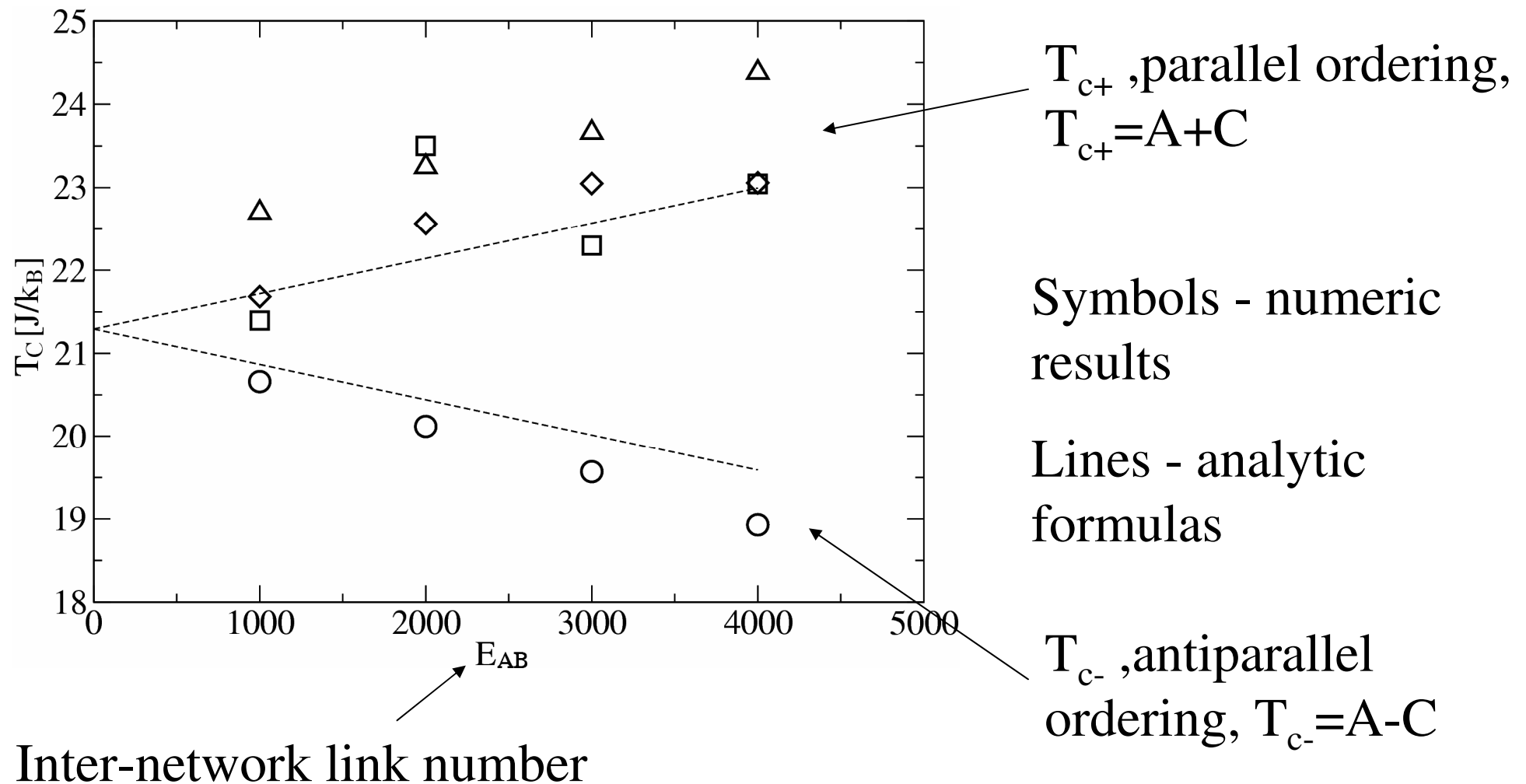
$$\langle s_{Ai} \rangle = \beta J_{AA} k_{AAi} \sum_j \frac{k_{Aj} \langle s_{Aj} \rangle}{E_{AA}} + \beta J_{BA} k_{ABi} \sum_j \frac{k_{BAj} \langle s_{Bj} \rangle}{E_{BA}}$$

$$\langle s_{Bi} \rangle = \beta J_{BB} k_{BBi} \sum_j \frac{k_{Bj} \langle s_{Bj} \rangle}{E_{BB}} + \beta J_{AB} k_{BAi} \sum_j \frac{k_{ABj} \langle s_{Aj} \rangle}{E_{AB}}$$

Similar to single scale-free network we introduce weighted spins $S_{A.}$, $S_{B.}$. Unfortunately we also need S_{AB} and S_{BA} .

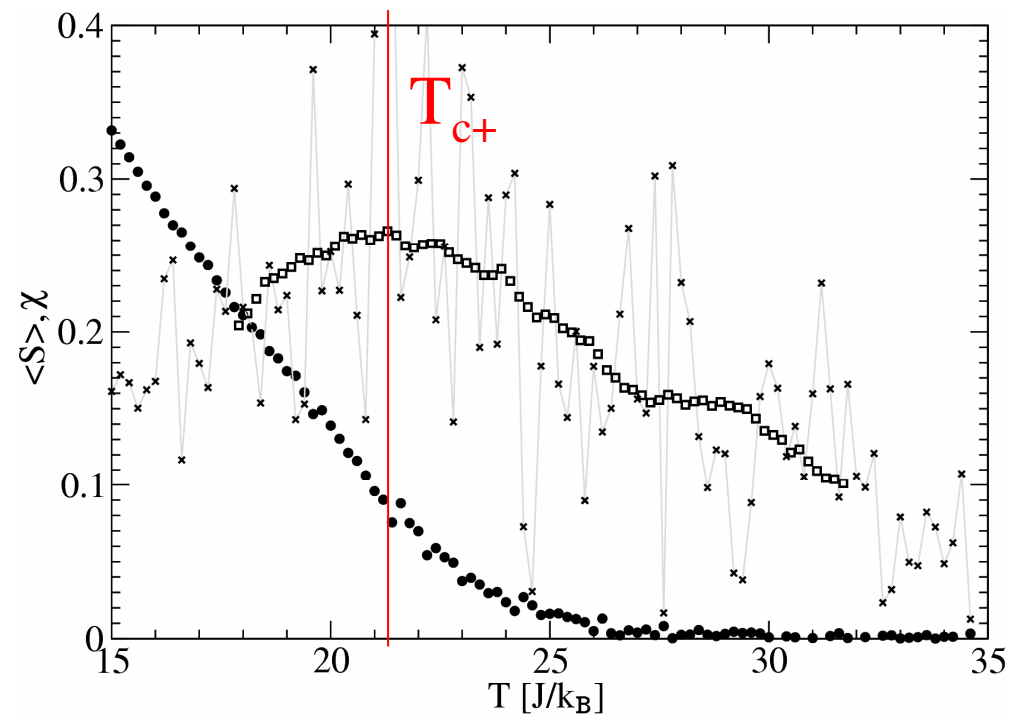
Numerical results

Numerical simulations for two same B-A ($N=5000$, $\langle k \rangle=10$, $D=0$) and various number of inter-network connections:



Numeric results

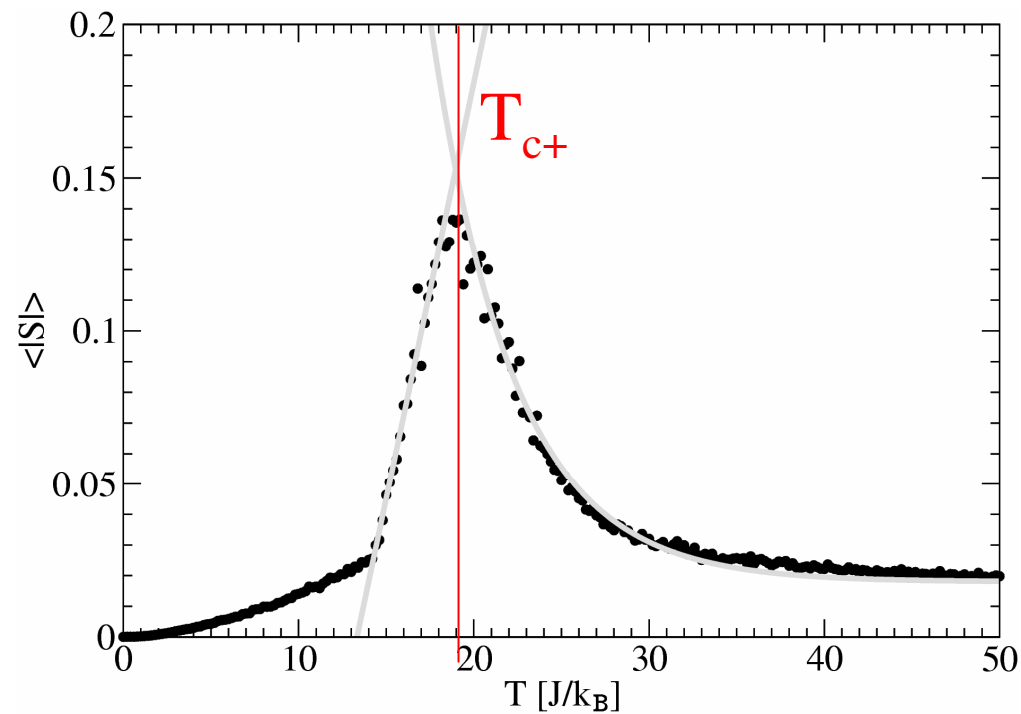
We have performed numeric simulations to find the critical temperatures T_{c+} and T_{c-} .



To find the critical temperature T_{c+} we calculate numerically the susceptibility $\chi = S_h - S_0$. Due to very highly fluctuating nature we make 30-point running average and fit parabolic curve.

Numeric results

We have performed numeric simulations to find the critical temperatures T_{c+} and T_{c-} .



Plot the total spin starting from antiparallel ordered state as a function of temperature.