WYDZIAŁ FIZYKI
POLITECHNIKA WARSZAWSKA

Voter model on Sierpinski fractals

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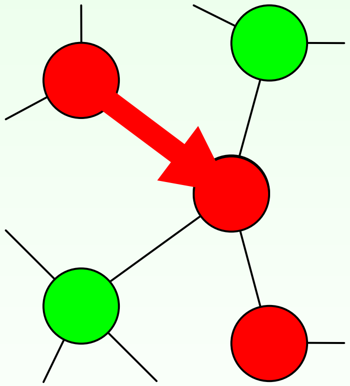
Plan

- What exactly is voter model ?
- How it behaves in regular lattices ?
- Fractal lattices: Sierpinski Gasket and Carpet
- The results
- And what can we conclude from them

Voter model

An extremely simple model of opinion formation.

Nodes in a network – agents. Each agent *i* has binary opinion σ_i



🟢 = ↓ = **-1**

🔴 = ↑ = **+1**

Rule: Assume state of one of neighbors (random choice)

Asynchronous dynamics :

We choose the node to update at random.

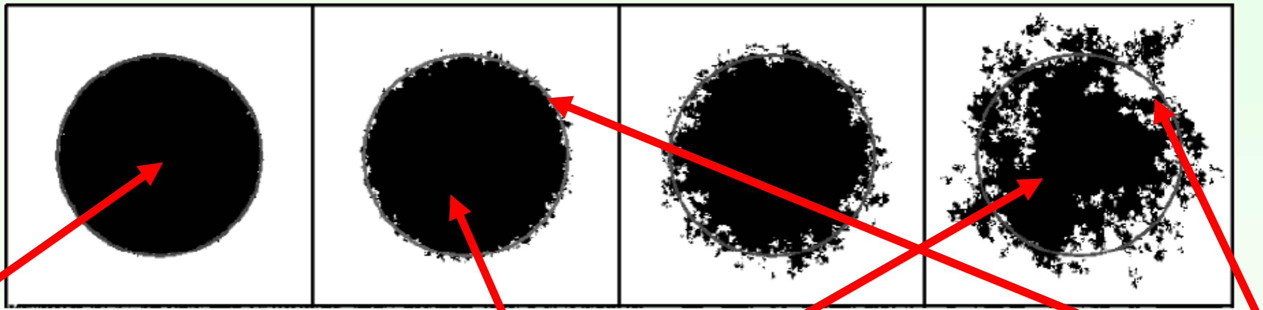
1 time step = N node updates (N-network size)

Voter model

Voter model has several general properties:

An example of voter dynamics on 2D square lattice

“Critical Coarsening without Surface Tension: The Universality Class of the Voter Model”, Ivan Dornic, Hugues Chaté, Jérôme Chave, and Haye Hinrichsen, Phys. Rev. Lett. 87, 045701 (2001)



No temperature-type noise : “white” noise cannot appear inside “black” domain

Conservation of spin: the “black” domain size doesn’t shrink or grow in thermodynamic limit

No surface tension: the curvature of surface doesn’t determine how it changes

Voter model

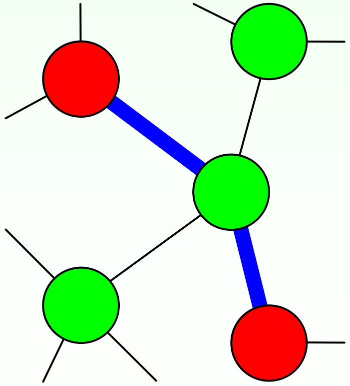
Dynamics

Ordering

The system coarsens, creating large domains of same state

Disorder

The system is in disordered state, without any large domains and minimal correlation lengths



Interfaces - links between nodes in different state.

A measure of system order.

Interface density:

$$\rho = \frac{\sum_{i=1}^{N} \sum_{j \in K_i} \frac{1 - \sigma_i \sigma_j}{2}}{\sum_{i=1}^{N} k_i}$$

Number of directed interfaces

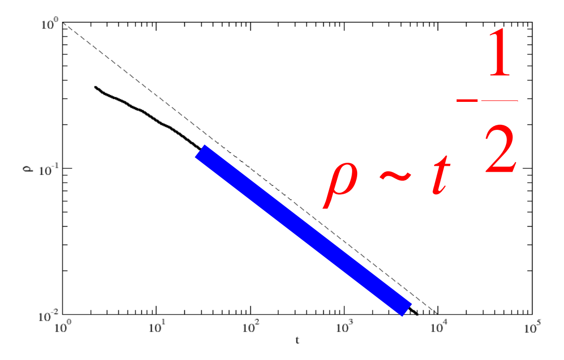
Number of total directed links

Voter model in regular lattices

Dynamics depend on network dimensionality.

1D

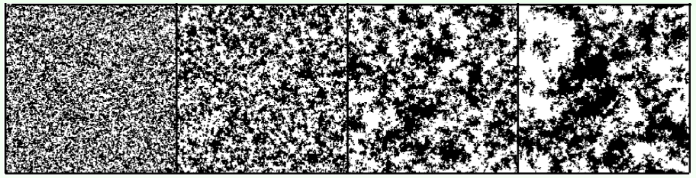
Ordering



$$\rho \sim t^{-\frac{1}{2}}$$

2D

Critical Ordering



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$$\rho \sim (\log t)^{-1}$$

3+D

Disorder

The system never orders, remaining in disordered state

$$\rho \sim const.$$

Red forumlas for interface density are **large time limit** analytic calculations by Frachenbourg and Krapivsky.

L.Frachenbourg, P.L.Krapivsky, "Exact results for kinetics of catalytic reactions", Phys. Rev. E 53, 3009 (1996)

Voter model in regular lattices

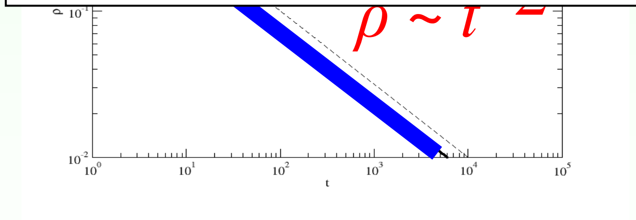
Dynamics depend on network dimensionality.

1D

2D

3+D

Fluctuations in finite systems always cause ordering of the system, with **exponential decay** of number of interfaces when the statistics include large amount of networks. The characteristic time of decay depends on the network size.



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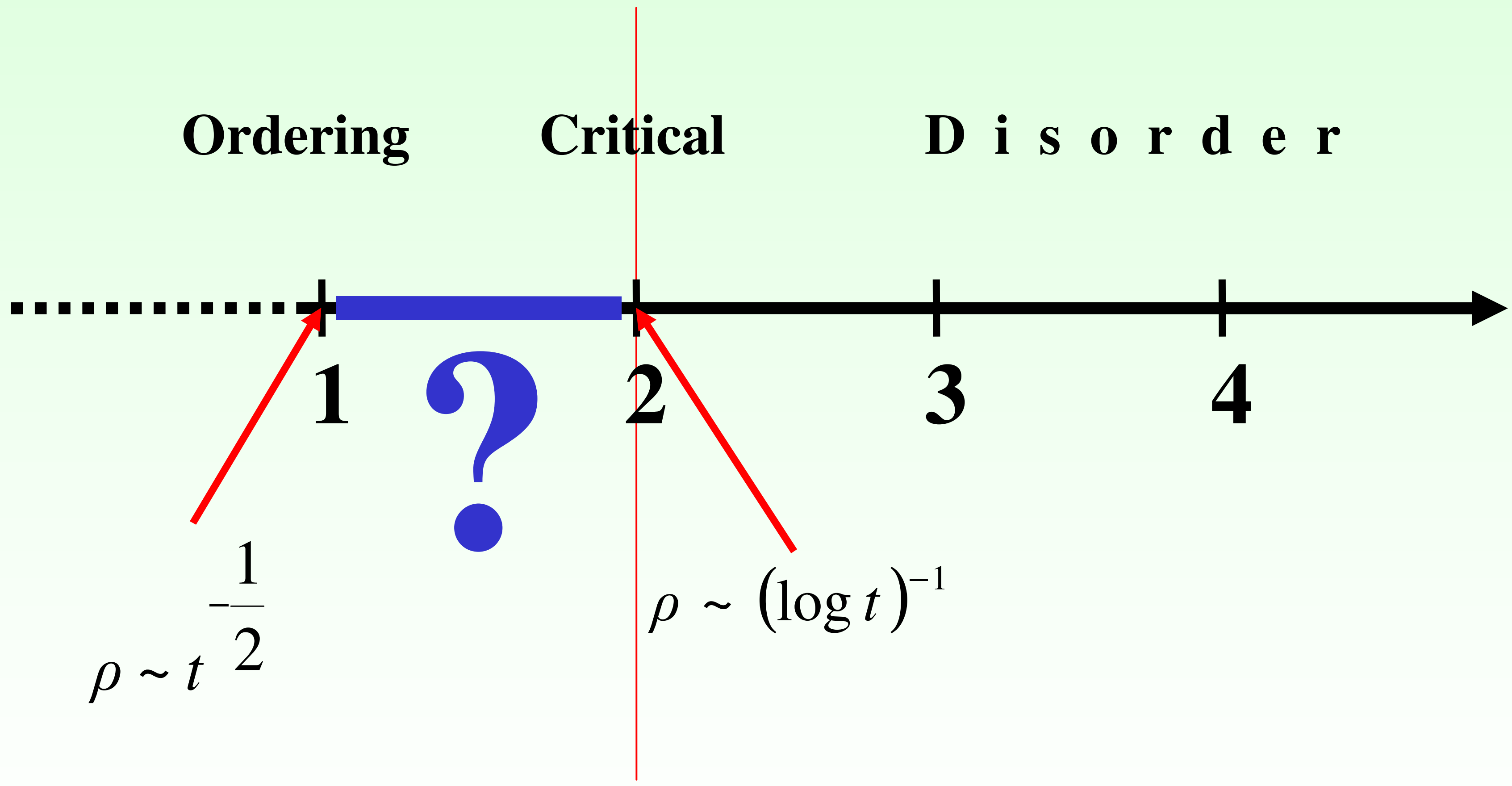
The Question



Ordering **Critical** **Disorder**

1 **?** **2** **3** **4**

$\rho \sim t^{-\frac{1}{2}}$

$\rho \sim (\log t)^{-1}$

How the voter model approaches the critical point ?

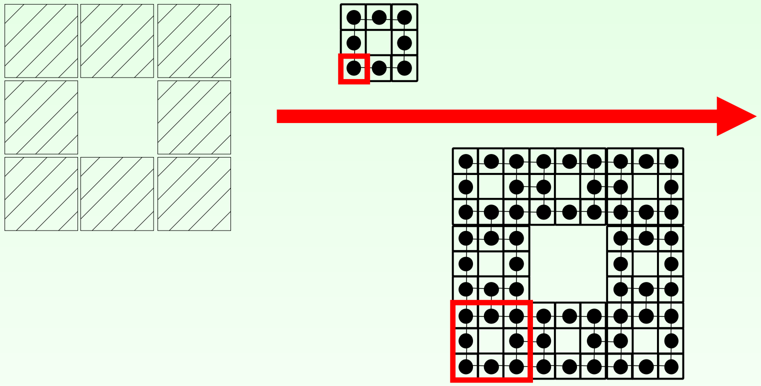
Does it behave differently on fractals than on regular lattices ?

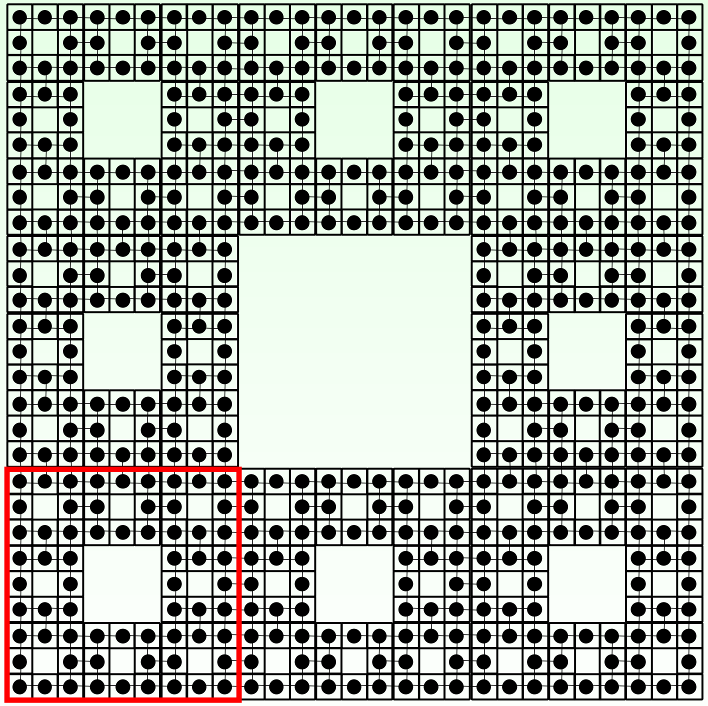
Sierpinski fractals

Sierpinski carpet

Fractal dimension $D = \ln 8 / \ln 3 \approx 1.893$

Basic fractal pattern





Networks obtained in previous step are arranged into basic pattern creating next step fractal network.

3rd step of fractal creation

Higher steps of fractals creation

Ramification:

the number of connections you must cut in order to separate arbitrarily large part of infinite network.

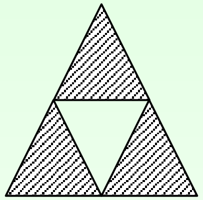
The number of links to cut goes to infinity - Ramification $R = +\infty$

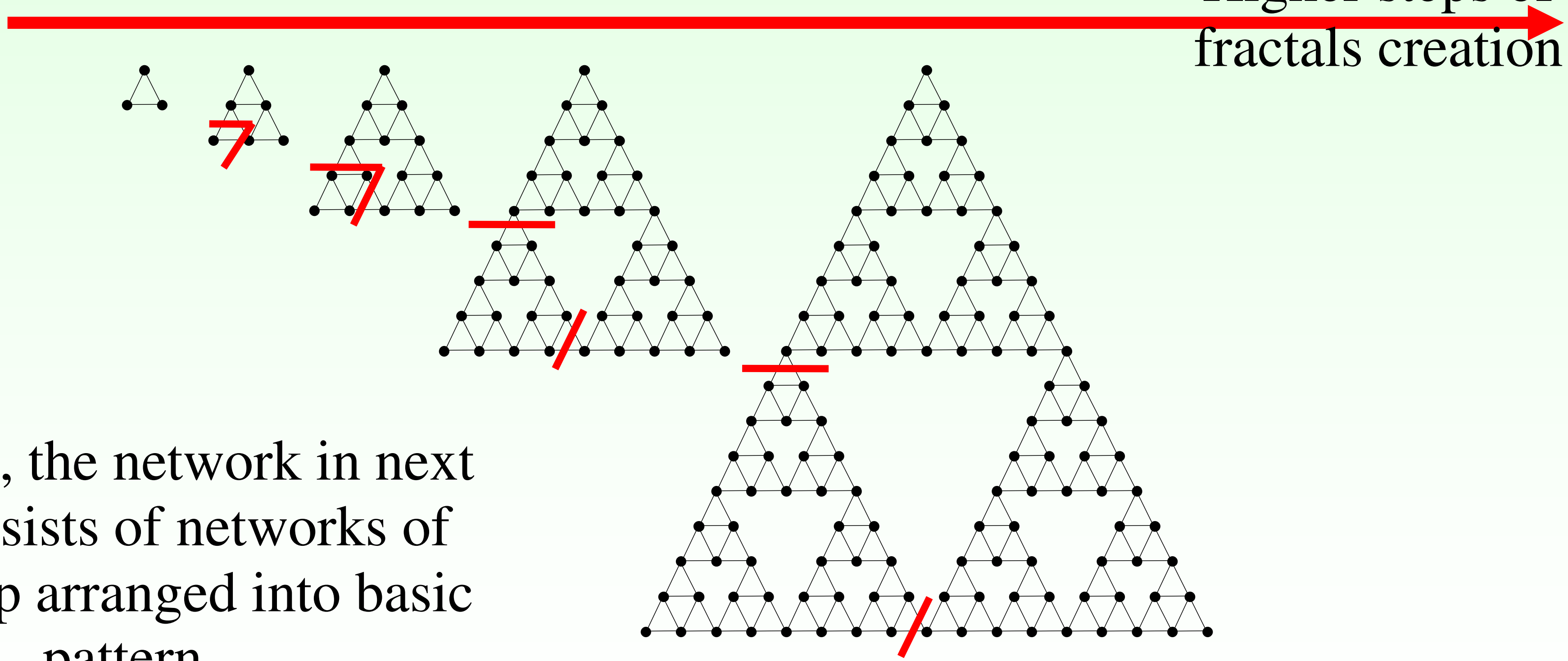
Sierpinski fractals

Sierpinski gasket

Fractal dimension $D = \ln 3 / \ln 2 \approx 1.585$

Basic fractal pattern





4th step of fractal creation

Similarly, the network in next step consists of networks of given step arranged into basic pattern.

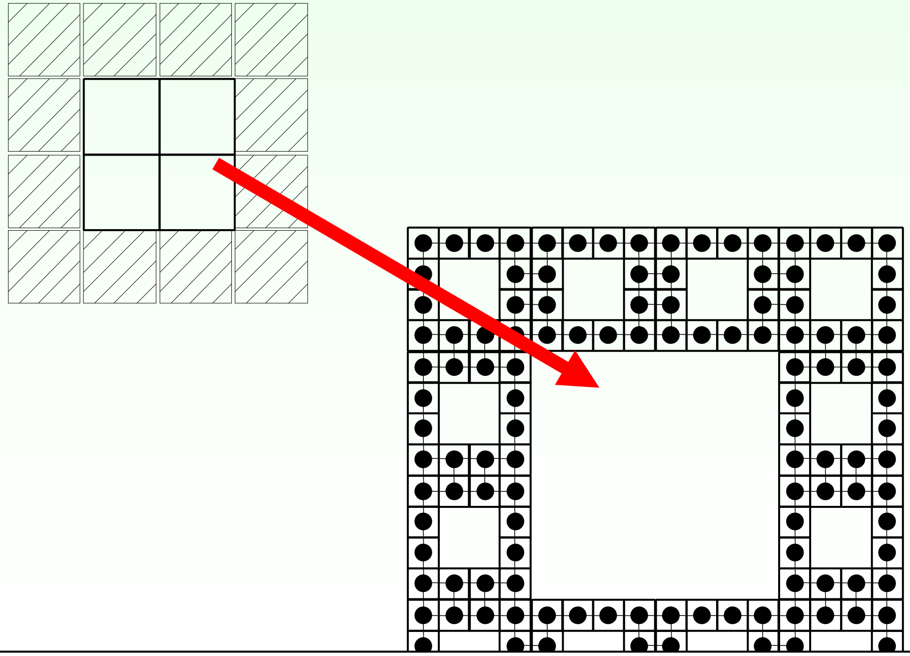
The number of links is constant - Ramification $R = 4$

Sierpinski fractals

Generalized Sierpinski carpet

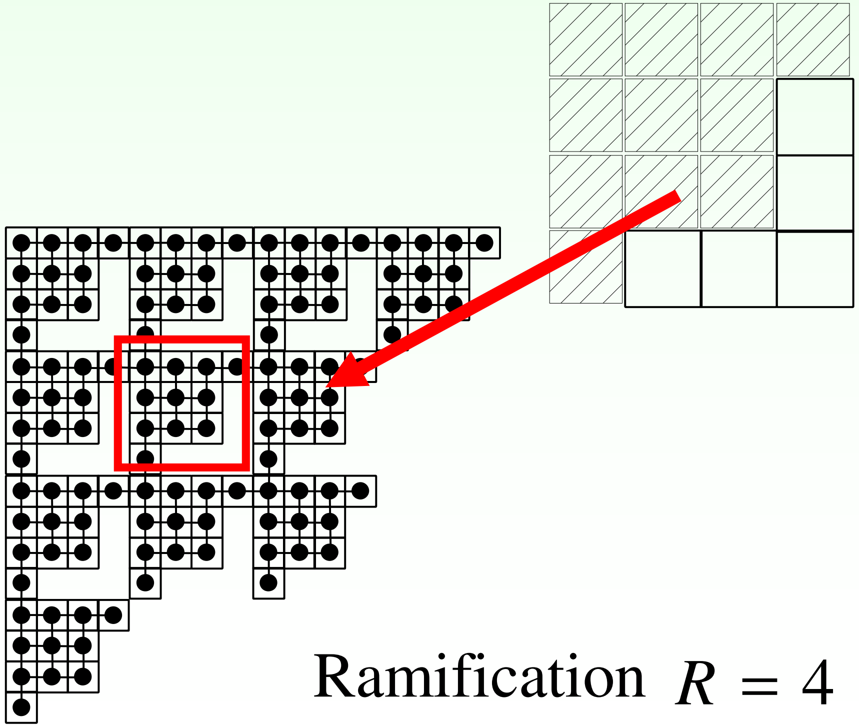
The model of sierpinski carpet can be easily generalized for any square basic pattern...

Allowing various fractal dimensions...



Fractal dimension $D = \ln 12 / \ln 4 \approx 1.792$

...or finite ramification

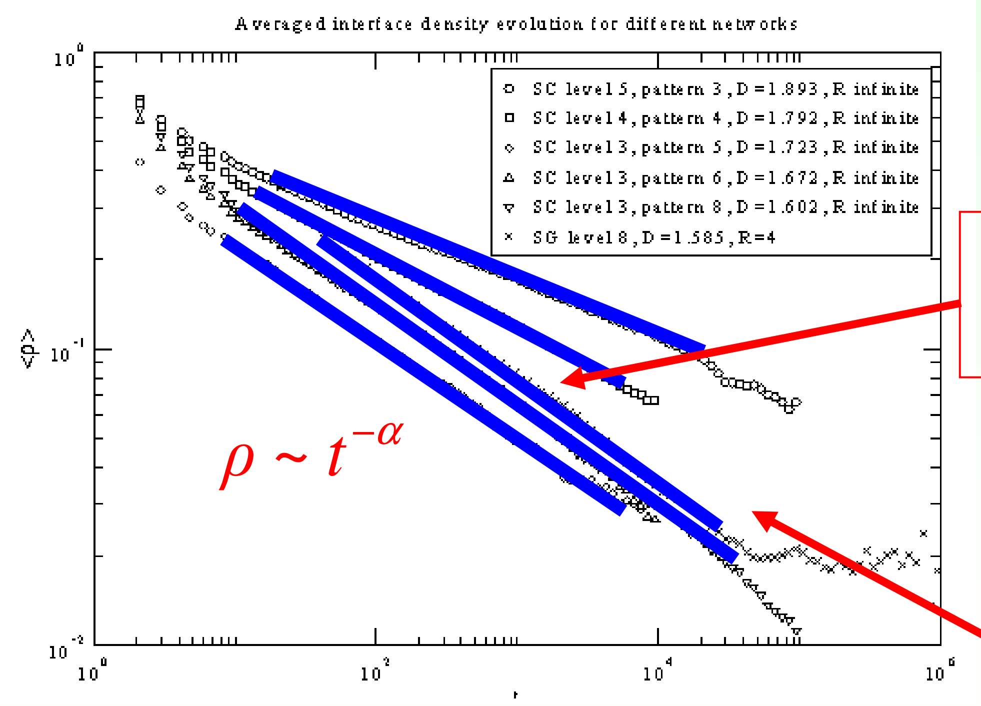


Ramification $R = 4$

Results

Evolution of interface density in time - the system orders, as expected below critical dimensionality.

Sierpinski carpet fractals with various fractal dimensions. The system orders similarly to one-dimensional regular lattice, but with different exponent.



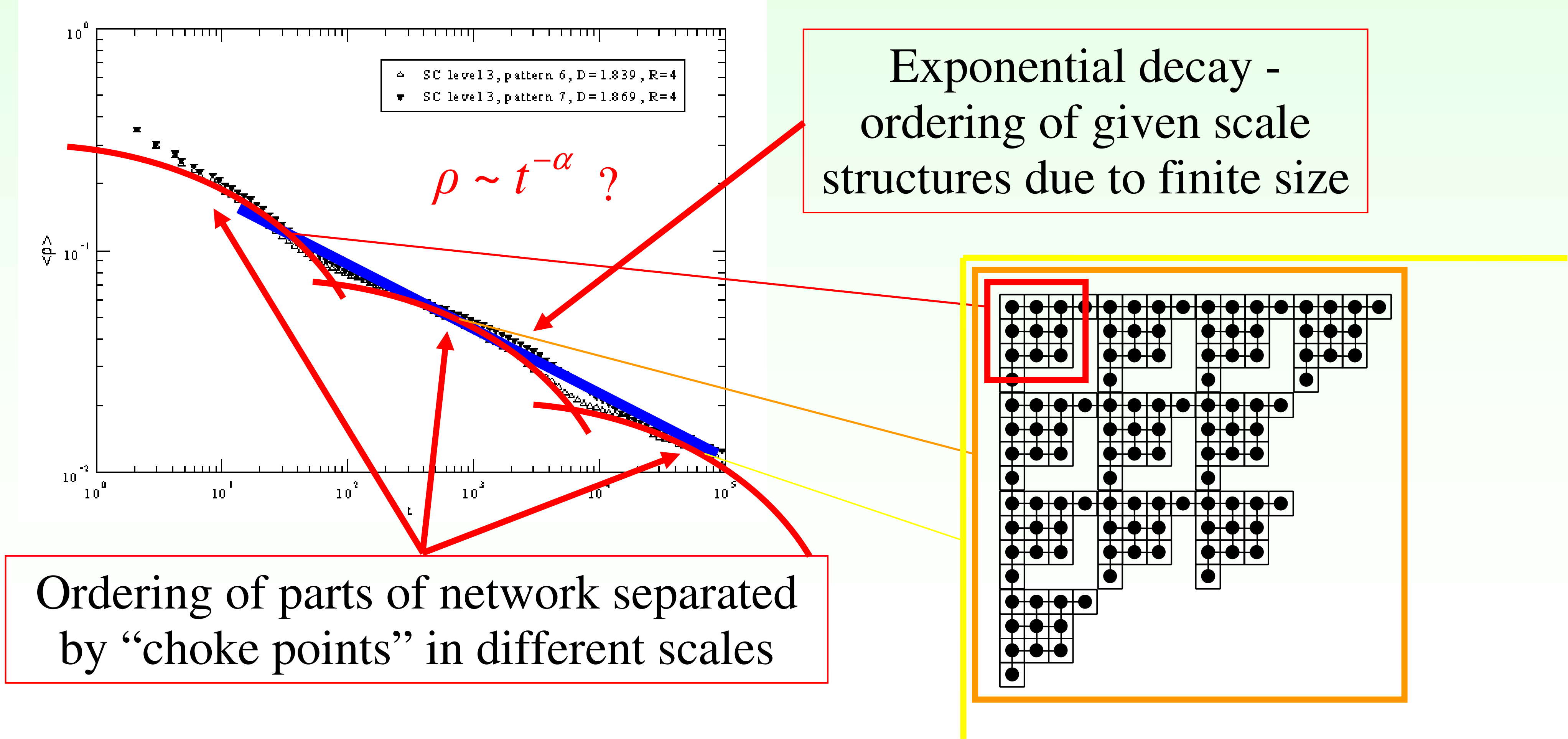
$\rho \sim t^{-\alpha}$

Voter model works just like in regular lattice but with fractional dimension

All sierpinski carpets here have infinite ramification

Results

Sierpinski carpets with finite ramification. The dynamics in that case may appear to be the same, but the presence of oscillations reveals different ordering mechanism.



Exponential decay - ordering of given scale structures due to finite size

Ordering of parts of network separated by “choke points” in different scales

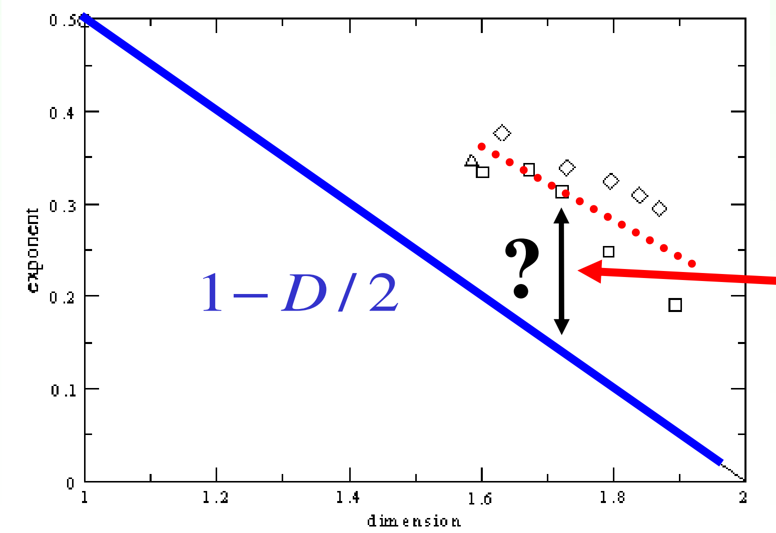
Results

Ordering exponent dependence on fractal dimension. The paper by Frachenbourg and Krapivsky gives analytic solutions for the ordering processes. Despite being calculated for square lattices, the final results are given for D<2, D=2 and D>2. For D<2, the result is following

$$\rho \sim t^{-1+D/2}$$

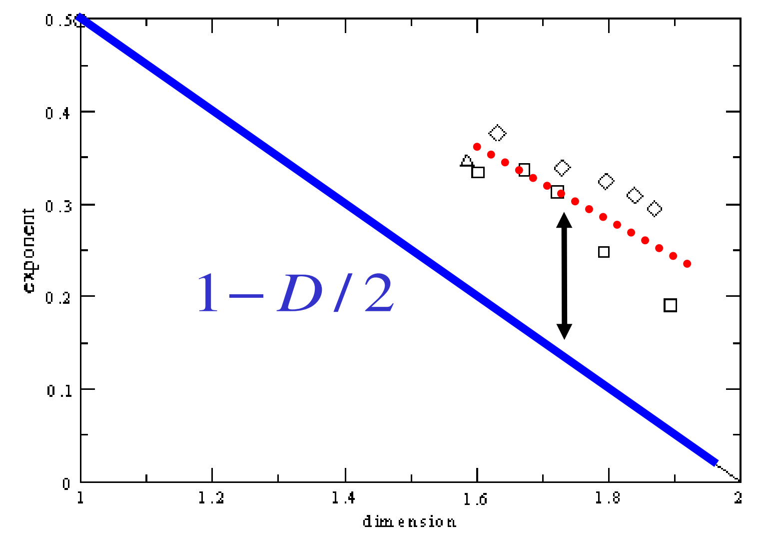
L.Frachenbourg, P.L.Krapivsky, "Exact results for kinetics of catalytic reactions", Phys. Rev. E 53, 3009 (1996)

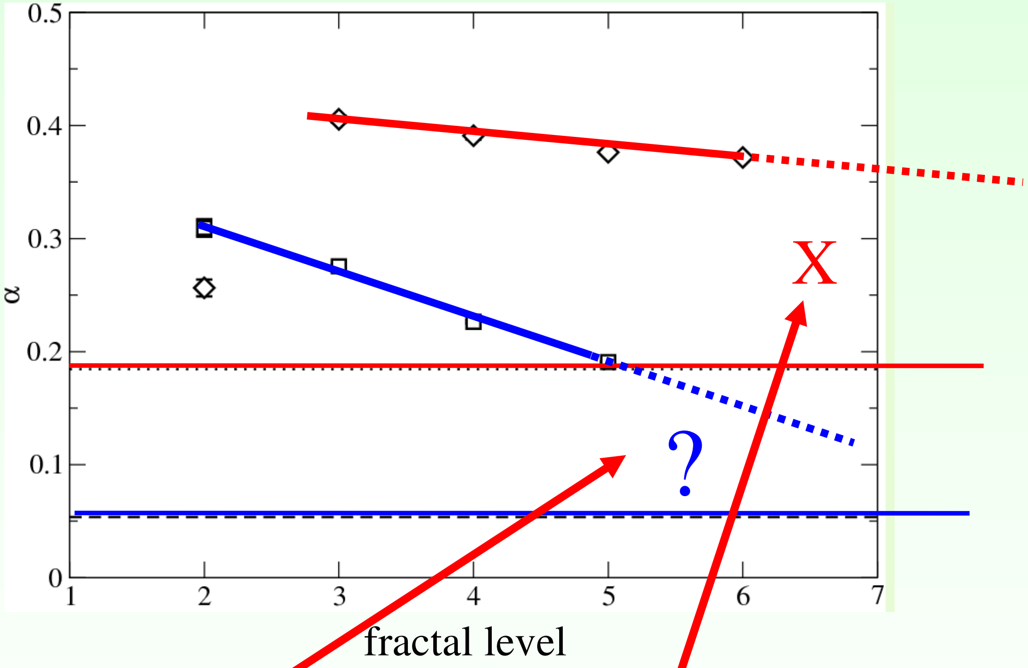
We check if this solution holds true for fractional dimensions.



The possible reason for disparity is the fact that all our fractal lattices were of course finite, as well as simulation time. The analytic results by Krapivsky are exact for infinite lattice and time.

Results





Infinite ramification: the exponent approaches theoretical value as fractal size increases – possible agreement

Finite ramification: the exponent is different consistently

Conclusions

- The voter model dynamics in fractal lattices are similar to regular lattices provided the ramification is infinite
- For fractals with finite ramification, the ordering mechanism is completely different, displaying log-periodic behaviour
- The analytic solutions for regular lattices cannot be extended to fractal lattices and still produce accurate results