Competing spreading processes on networks –

even a few localized edges give you the winning edge.

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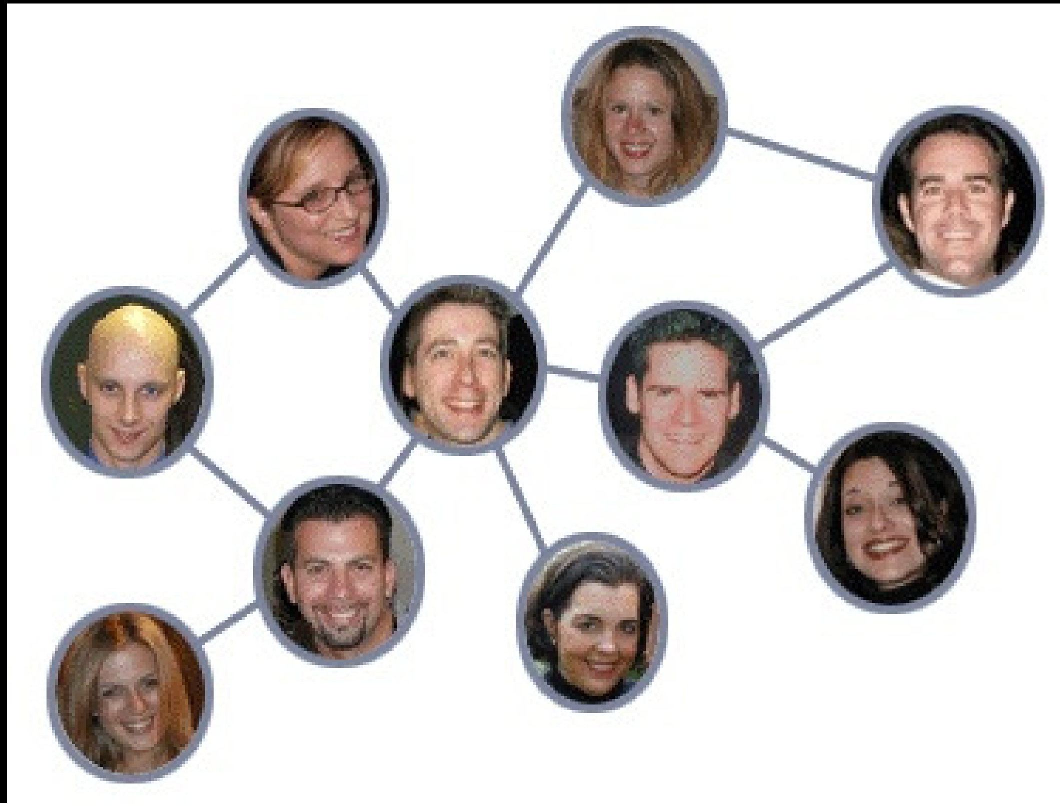
Motivation

Competing spreading processes on networks

- Example: consider a disease in a network which has the ability to develop an immunizing agent (antivirus) and instigate its propagation.

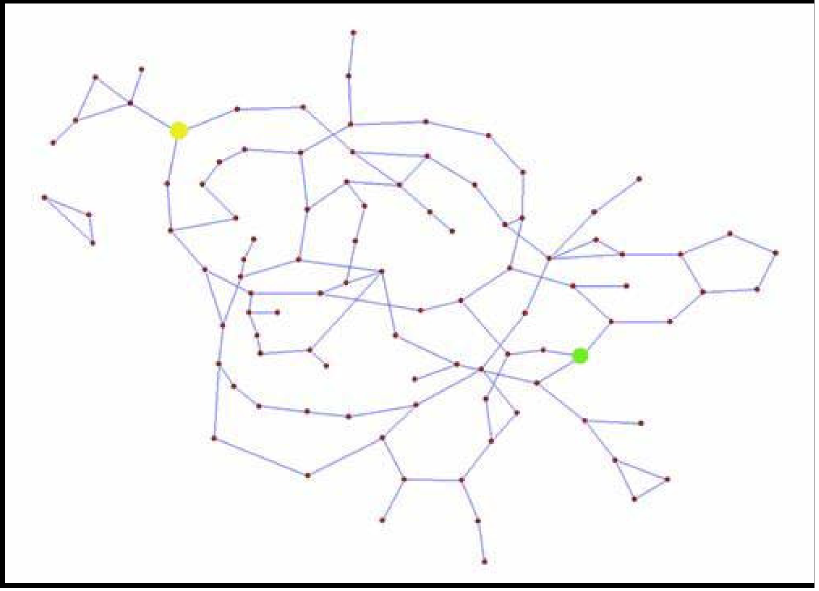


A one minute layout..

- Motivation
- Previous work - Goldenberg et al. 2005, inspiring the idea of adding extra edges solely for the antivirus transmission.
- Our method –
  - Adding edges in a natural and local manner
  - More accurate tools
    - Distances between pairs of nodes in random graphs (van der Hofstad et al 2005).
    - Non-homogenous recurrence relation for the distances after the addition.
  - Uncovering the “growth rate” effect, the cause of the antivirus triumph.
- Analysis and simulations

Simplest setting

- A network of N individuals.
- At time step t = 0 a (randomly chosen) node is infected
  - In each following time step the neighbours of infected nodes are being infected (it is possible to consider rate/probability of transmission).
- When specific (randomly pre-assigned) nodes get infected, they develop an antivirus which start immunizing the network similarly to the virus.



Simplest setting – (non) crucial details

- Does the “yellow” nodes infect their susceptible neighbours with the virus?
  - In our model: yes. Much more challenging.
- Can the antivirus “backtrack” trough infected nodes?
  - In our model: yes. More suitable for computer networks/viruses.
- How many “yellow” nodes are there?
  - We shell see that it is nice to have it >> log(N) , but still o(N).

Simplest setting – helping the antivirus

- We can add a fraction, q, of special edges
- Through the special edges, only the antivirus can be transmitted.
- The objective is to minimize q, while securing the advantage of the antivirus.

We have just described a system for arbitrary networks, the next natural step is to study it on a random graph model (such a ER, MR)!

Notations:

- We need some notation:
  - Let N , the number of nodes, be fixed.
  - $A_t(N)$ is the total number of immunized by time step t
  - $V_t(N)$ is the total number of infected by time step t
  - l_t is the number of infected exactly at time step t
  - α is the "branching factor" related to the first and second moments of the degree distribution.

Brief analysis..

$l_{t+1} = l_t \cdot \alpha$. If α is larger than 1 we get an exponential growth. If N is large enough so that finite-size effects are irrelevant we get

$$V_t(N) = (1 + \alpha + \alpha^2 + \cdots + \alpha^t) = \frac{\alpha^{t+1} - 1}{\alpha - 1}.$$

Let us turn to the immunized cluster(s). Define $A_t(N)$ to be the aggregate size of the immunized clusters at time step t as a function of N. Given a relative edge addition q and an average node degree m, the expected number of immune specific edges is qm, and each may initiate an immunized cluster. Once started, the immunized clusters also grow with ratio α. Thus, A_t when N is large enough is

$$A_t(N) = qm[(t-1) \cdot \alpha^{t-2} + (t-2) \cdot \alpha^{t-3} + \cdots + 1],$$

which can be compacted to

$$A_t(N) = qm\left[\frac{t\alpha^{t-1}}{\alpha - 1} - \frac{\alpha^t - 1}{(\alpha - 1)^2}\right].$$

The ratio of the size of the virus cluster to that of the immunized clusters is

$$\frac{V_t(N)}{A_t(N)} = \frac{1}{qm}\left[\frac{(\alpha^{t+1} - 1)(\alpha - 1)}{(t-1)\alpha^t - t\alpha^{t-1} + 1}\right],$$

Possible improvements

- Adding edges in a natural and local manner.
- α_{AV} is > α_V
  – Even if only by a small amount, ε, the geometrical progression ensure a win.
- t can not grow arbitrarily!
  – But this seems to spoil also our previous scheme.. (we need a way to go round this problem)

Our approach

Adding edges

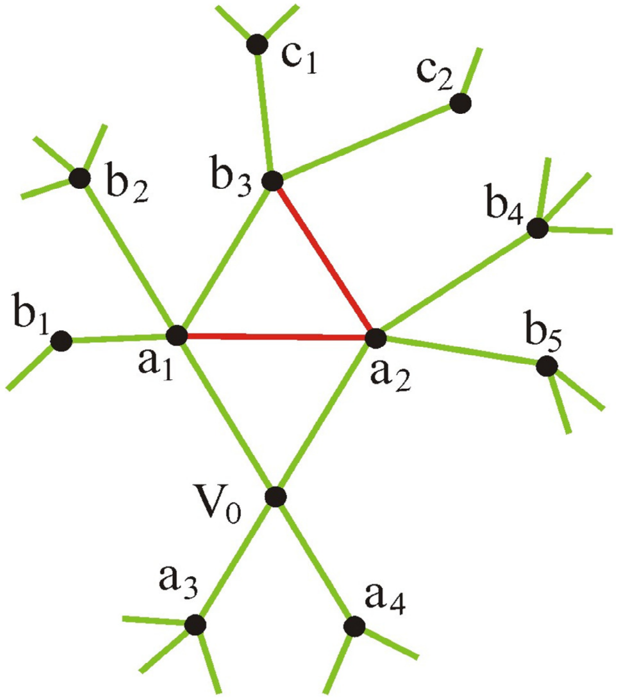
- Connect each pair of nodes at a distance two with probability C.
  - C can be very small (or even a slowly decreasing function C(N)).

- Local, decentralized and natural.
- Example:

  the use of the "reply all" option instead of "reply" in an e-mail correspondence, where a new connection is made between individuals that might have never been in contact before

Adding edges – important remark

- The increase in the average degree does not guarantee us a similar increase in the branching factor, because the network is highly clustered!
  - In contrast to the random (non-local) edge addition.



- Is it possible to use the increased growth rate and also deal with the fact that $t < N$?
- YES! Instead, we use a “dual” approach and study distances in the network.

the distances in random graphs – without addition of edges

- Starting at a certain node, we would expect most of the nodes in the graph to be in a distance of log(N) from it (where the base of log is the branching factor)
- "Dutch theorems" – this is actually concentrated! (finite variance)

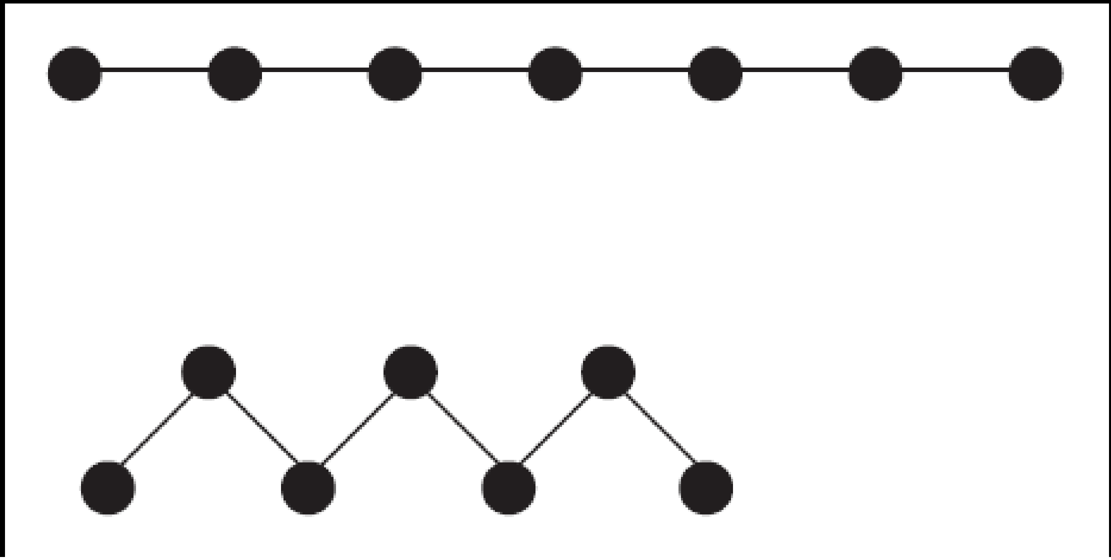
the distances in random graphs – without addition of edges

- As a corollary we get a "race track" at a length of log(N), winner takes it all.
- However, even when the antivirus start propagating a short while after the virus does – it loses.

the distances in random graphs – with addition of edges

- How does the length of the "track" decreases (on average) for the antivirus?
- Find h(n) , where n is the length without the new edges.

e.g. : n = 7

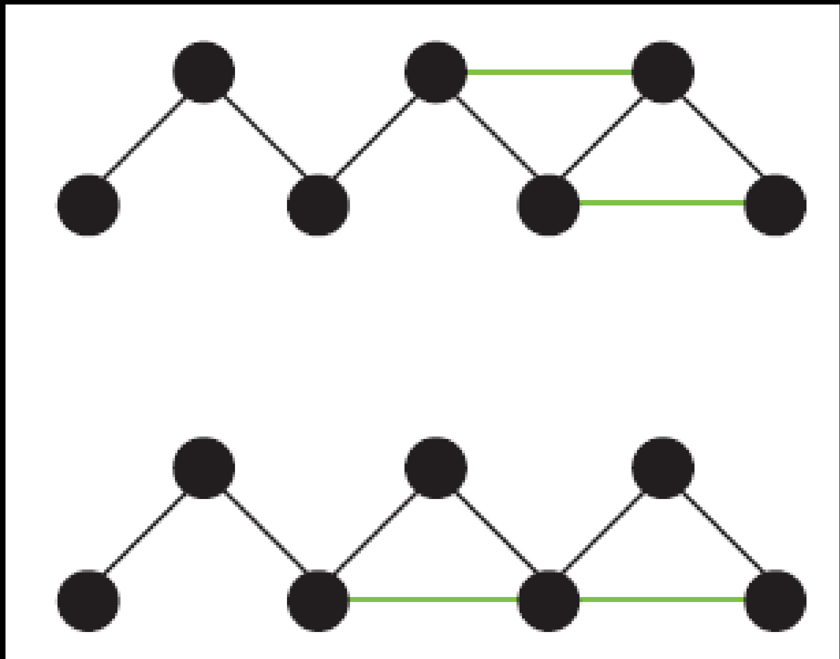


the distances in random graphs – with addition of edges

- Correlated scenarios..

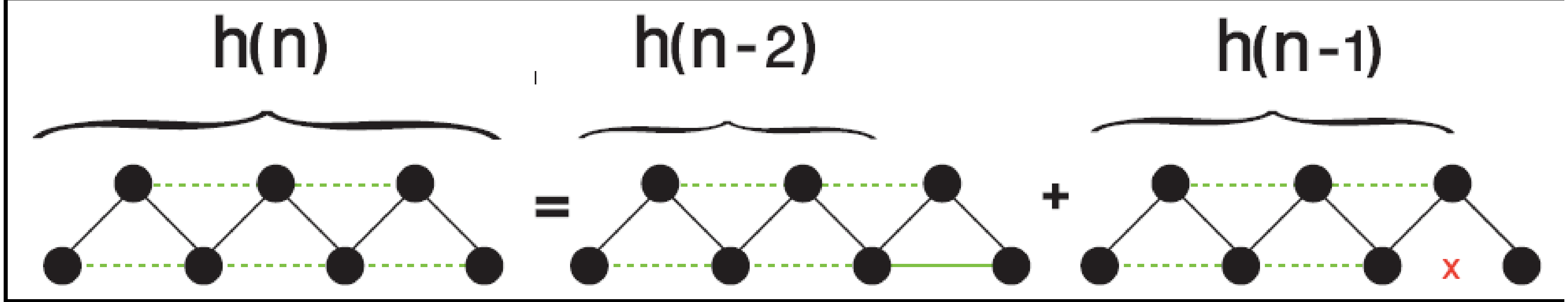
e.g. : h(n) = 5

e.g. : h(n) = 4



the distances in random graphs – with addition of edges

- Last edge considerations



h(n) = C (h(n-2) + 1) + (1-C)(h(n-1) + 1)

the distances in random graphs – with addition of edges

- Recurrence relation:

$$\begin{cases} h(n) = (1-C)\, h(n-1) + C\, h(n-2) + 1 \\ h(0) = 0\,,\ h(1) = 1 \qquad \text{initial conditions} \end{cases}$$

the distances in random graphs – with addition of edges

Easiest method..

open MATHEMATICA and type:

RSolve[{h[n+2]==(1-C) h[n+1]+C h[n]+ 1,h[0]==0,h[1]==1},h[n],n]

ENTER and get:

$$\left\{\left\{h[n] \to \frac{-(-C)^n\left(\left(-\frac{1}{C}\right)^n + C\right) + (1+C)\,(1+n)}{(1+C)^2}\right\}\right\}$$

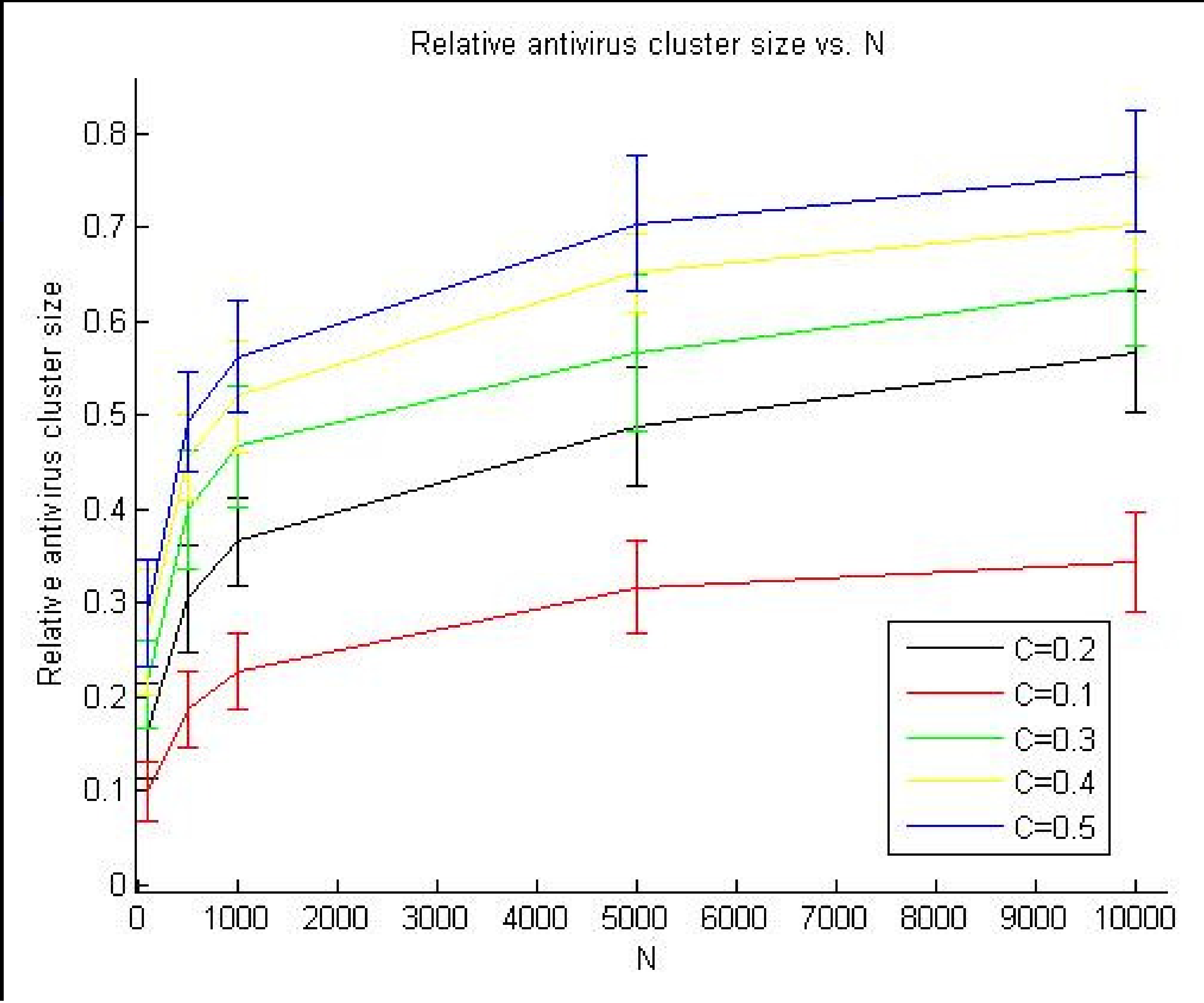
Which → n / (1+C) as n → ◎

Conclusion:
the addition decreases the distances by a substantial, non-finite amount

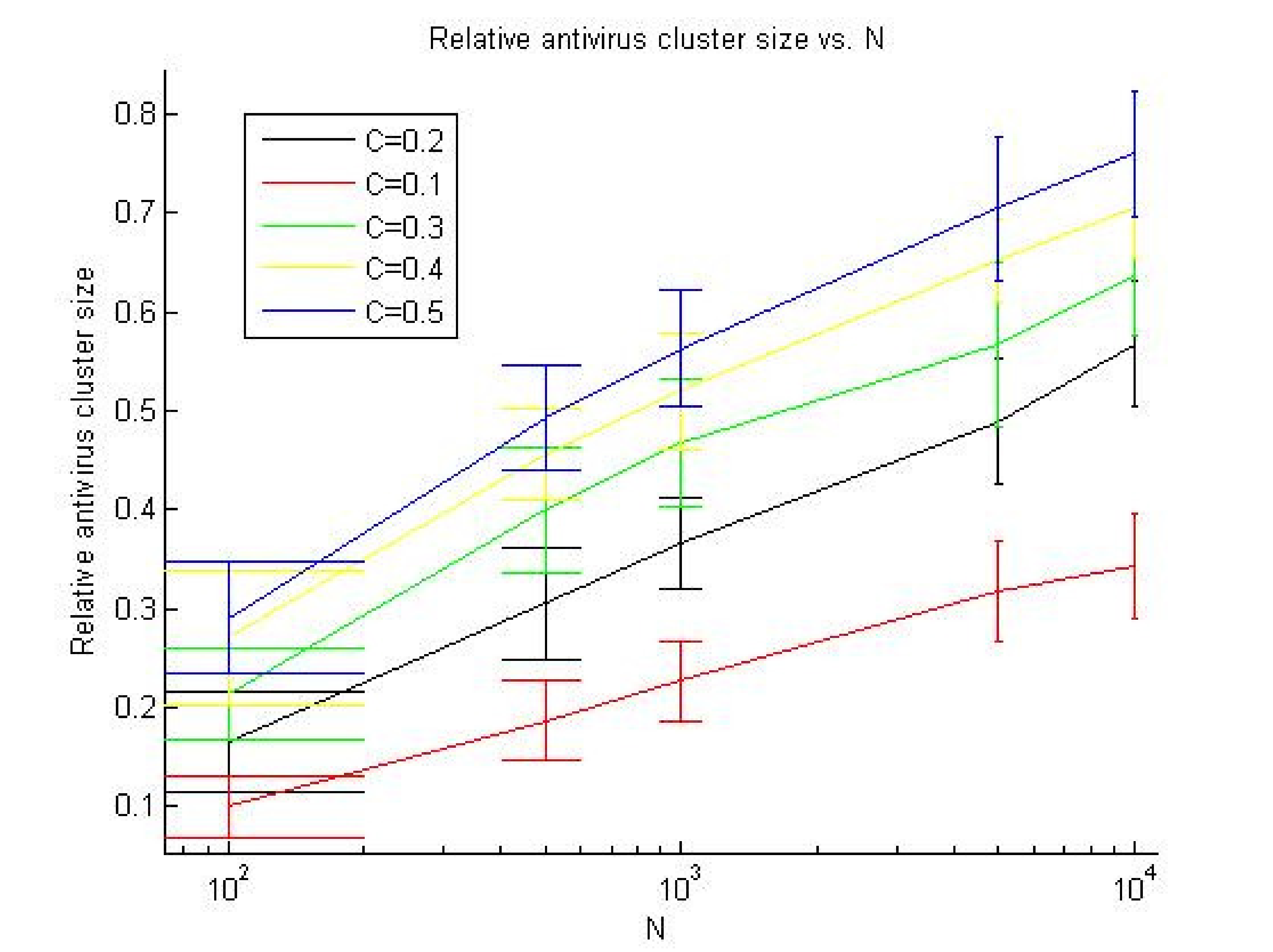
$$h = \log(N)/(1+C)$$

simulations

simulations



simulations



Summary of the new results

1. We studied the effect of a new naturally feasible local decentralized edge addition.
   - 1/(1+C) effect.
2. Our first result was used to explain the victory of the AV.
3. .. As demonstrated by simulations.

Future work

Finding out the finite size effects.

Acknowledgments..

- Mina Teicher
- The group.. and especially Yaki