Color-Swap Models for Non-growing Scale-free Networks

Tomas Hruz, Madhuresh Agrawal and Michal Natora

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The Overall Picture
The edge-colored class of models

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Conclusion

Almost every second picture of complex network you have seen on this conference has colored edges.

The researchers start to investigate the relations between different networks constituted by different types of links.

Example: nodes are companies, blue connection means a business, red means a board of directors relation.

The rules which relate the different networks (colors) on the microscopic level.

Colors can expand our modeling possibilities:
- orientation
- +/-
- connection strength (capacity, flow)
- color
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Biological Networks

- Interdisciplinary ETH project Genevestigator
  - Biology
  - Computer Science
  - Physics

- Genevestigator contains large high quality database of gene expression data. We have two complex network models in Genevestigator
  - Biochemical reaction pathways. The nodes represent biochemical reactions and the edges the reaction inputs and outputs. We have 5 types of edges (edge colors) for metabolites, enzymes, stimuli, cofactors and effects.
  - Gene correlation (control) network. The nodes are genes, and the edges represent regulatory relations between the genes.

- We are also starting a cooperation with mobile operators to investigate the structure and evolution of the mobile call networks
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Non-growing networks

We focus on non-growing networks
- The main interest is not how the networks were created but more how they operate and change mainly in the sense of rewiring the links between the nodes.
- We measured our metabolic networks and they are scale free.
- The size of the networks is moderate: metabolic pathways has about 4000 nodes, 12000 edges and gene regulation network has about $10^4$ nodes.
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The model we are looking for must have the following features:

1. It must achieve a stable scale-free state.
2. The number of edges must be constant or bounded in a small interval.
3. The network should not condensate even if the size and the number of edges is moderate $10^3 - 10^4$.
4. There should not be structural constraints, like for example constant out-going degree.
5. The stochastic process must stay in the class of simple graphs - no multiple edges and self-loops.
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Color-swap model

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The color-swap model

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Edge-colored complete graph with \( \binom{N}{2} \) edges.

The simple model has three colors: black, blue and red.

Black edges: non-edges - no connection (not shown).

Blue edges: the real connections.

Red edges: potential connections.

The only allowed operation is an exchange of color between the edges.
The color-swap model

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Color-swap process

- Initial condition.
- Blue Rewiring Phase: Uniformly at random select two nodes $V_i$ and $V_j \neq V_i$. For each blue neighbor $V_k$ of $V_j$ swap the edge.
- Red Rewiring Phase: Select a vertex $V_l$ with a blue linear preference. Uniformly at random select a blue edge $E_b$ incident on it and swap it with an uniformly at random chosen red edge $E_r$. 

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(a) Initial condition.

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The degree distribution

Color-Swap

![Log-log plot of degree distribution](image)

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Simulation Results

Discussion

Parameters used: \( N = 100000, L_{\text{blue}} = L_{\text{red}} = 300000 \), thus \( L_{\text{blue}} \ll L_{\text{black}} \).

- Stable scale-free distribution, even for small networks.
- No structural constraints on in/out degree.
- Number of edges (blue, red) is constant.
- The process naturally stays in the class of simple graphs.
- There is a fluctuation on the degree of certain nodes.
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- Interest in small networks.
- Simple graphs in our applications.
- Study of simple graph constraints on the most simple process.
- Simple Graph Edge Selection Process - SGESP
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Simple Graph Edge Selection Process

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- Select a node \( v_l \) preferentially with probability \( \frac{f(k)}{N(f)} \)
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락스 - 벡터 
소도 - 노드 
심플 그래프 간선 선별 과정

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Hierarchy of Object Distributions

\[ P(k, k') \sim \frac{L(k, k')}{L} \]

- \( N(1) = 1, N(2) = 3, N(3) = 1 \)
- Left graph: \( L(1, 2) = 1, L(2, 2) = 1, L(2, 3) = 3 \), right graph: \( L(2, 2) = 2, L(1, 3) = 1, L(2, 3) = 2 \)
- \[ P(k) = \frac{\bar{k}}{k} \sum_{k'} P(k, k') \]
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Hierarchy of Object Distributions
Why we need more complex objects

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Why we need more complex objects

- \( N(1) = 2, N(2) = 2, N(3) = 2 \)
- \( L(1, 3) = 2, L(2, 2) = 1, L(2, 3) = 2, L(3, 3) = 1 \)
- but \( N_L(1, 3) = 3 \) and \( N_R(1, 3) = 4 \)
Hierarchy of Object Distributions

\[ P(k, k', k'') \sim W(k, k', k'') / W \]

- \( N(k), L(k, k') \) are equal
- Left graph:
  \[ W(2, 2, 3) = 2, \ W(2, 3, 3) = 2, \ W(1, 3, 3) = 2, \ W(1, 3, 1) = 1 \]
- and right graph:
  \[ W(2, 2, 3) = 2, \ W(2, 3, 3) = 2, \ W(1, 3, 3) = 2, \ W(1, 3, 2) = 2 \]
- \[ P(k, k') = \sqrt{2}LC_0 \frac{\sum_P(k, k', k'') + P(k'', k, k')}{k + k' - 2} \]
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- \( P(k, k') = \sqrt{2LC_0} \frac{\sum_{k''} P(k', k'', k'') + P(k'', k, k')}{k + k' - 2} \)
Hierarchy of Object Distributions

\[ P(k, k', k'') \sim \frac{W(k, k', k'')}{W} \]

1. **N(k), L(k, k') are equal**
2. **Left graph:**
   \[ W(2, 2, 3) = 2, \ W(2, 3, 3) = 2, \ W(1, 3, 3) = 2, \ W(1, 3, 1) = 1 \]
   
3. **Right graph:**
   \[ W(2, 2, 3) = 2, \ W(2, 3, 3) = 2, \ W(1, 3, 3) = 2, \ W(1, 3, 2) = 2 \]

4. **P(k, k')**
   \[ P(k, k') = \sqrt{2LC_0} \sum_{k''} \frac{P(k, k', k'') + P(k'', k, k')}{k + k' - 2} \]
Simple Graph Edge Selection Process

- Wedges are enough to analytically express the conditional flow in the process

\[ P(k', k'') \frac{f(k''')}{N(f(k))} (N(k''') - \delta_{k''',k'} - \delta_{k''',k''}) \times (1 - P_{k''',(k'',k')}) \]

- \( P(k', k'') \) is the probability that vertex \( V_i \) is of degree \( k' \) and it has a direct neighbor vertex \( V_j \) of degree \( k'' \)

- \( \frac{f(k''')}{N(f(k))} (N(k''') - \delta_{k''',k'} - \delta_{k''',k''}) \) is the probability that vertex \( V_i \), which is selected with a probability proportional to \( f(k) \), is of degree \( k'''' \) and is not a part of the edge \( E_i \)

- \( (1 - P_{k''',(k'',k')}) \) is equal to the probability that there is no edge between \( V_i \) and \( V_j \)

\[ P_{k,(k',k'')} = \frac{2W \cdot P(k,k',k'')}{(N(k)-\delta_{k',k'}-\delta_{k,k''})L(k',k'')} \]
Simple Graph Edge Selection Process

**Configuration term**

- Wedges are enough to analytically express the conditional flow in the process

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- \( P(k', k'') \) is the probability that vertex \( V_i \) is of degree \( k' \) and it has a direct neighbor vertex \( V_j \) of degree \( k'' \)

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Wedges are enough to analytically express the conditional flow in the process

$P(k', k'') \frac{f(k''')}{N(f(k))} (N(k''') - \delta_{k''',k'} - \delta_{k''',k''}) \times (1 - P_{k''',(k'',k')})$

- $P(k', k'')$ is the probability that vertex $V_i$ is of degree $k'$ and it has a direct neighbor vertex $V_j$ of degree $k''$
- $\frac{f(k''')}{N(f(k))} (N(k''') - \delta_{k''',k'} - \delta_{k''',k''})$ is the probability that vertex $V_l$, which is selected with a probability proportional to $f(k)$, is of degree $k'''$ and is not a part of the edge $E_i$
- $(1 - P_{k''',(k'',k')})$ is equal to the probability that there is no edge between $V_i$ and $V_j$

$P_{k,(k',k'')} = \frac{2W \cdot P(k,k',k'')}{(N(k)-\delta_{k,k'}-\delta_{k,k''})L(k',k'')}$
Simple Graph Edge Selection Process

Configuration term

- Wedges are enough to analytically express the conditional flow in the process

\[ P(k', k'') \frac{f(k''')}{N\langle f(k) \rangle} (N(k''') - \delta_{k''', k'} - \delta_{k''', k''}) \times (1 - P_{k''', (k'', k')}) \]

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- \( (1 - P_{k''', (k'', k')}) \) is equal to the probability that there is no edge between \( V_i \) and \( V_j \)

\[ P_{k, (k', k'')} = \frac{2W \cdot P(k, k', k'')}{(N(k) - \delta_{k, k'} - \delta_{k, k''})L(k', k'')} \]
Simple Graph Edge Selection Process

Configuration term

- Wedges are enough to analytically express the conditional flow in the process

- \( P(k', k'') \cdot \frac{f(k''')}{N(f(k))} \left( N(k''') - \delta_{k''', k'} - \delta_{k''', k''} \right) \times \left( 1 - P_{k''', (k'', k')} \right) \)
  - \( P(k', k'') \) is the probability that vertex \( V_i \) is of degree \( k' \) and it has a direct neighbor vertex \( V_j \) of degree \( k'' \)
  - \( \frac{f(k''')}{N(f(k))} \left( N(k''') - \delta_{k''', k'} - \delta_{k''', k''} \right) \) is the probability that vertex \( V_{l} \), which is selected with a probability proportional to \( f(k) \), is of degree \( k'''' \) and is not a part of the edge \( E_{i} \)
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- \( P_{k,(k',k'')} = \frac{2W \cdot P(k,k',k'')}{(N(k) - \delta_{k,k'} - \delta_{k,k''})L(k',k'')} \)
Simple Graph Edge Selection Process

Master equation

\[ N(k, t + 1) = \]
\[ N(k) - \sum_{k', k''} P(k', k'') \frac{f(k)(N(k) - \delta_{k, k'} - \delta_{k', k''})}{N\langle f \rangle} (1 - P_{k,(k',k'')}) + \]
\[ \sum_{k', k''} P(k', k'') \frac{f(k-1)(N(k-1) - \delta_{k-1, k'} - \delta_{k', k''})}{N\langle f \rangle} (1 - P_{k-1,(k',k'')}) - \]
\[ \sum_{k', k'''} P(k', k) \frac{f(k''')(N(k''') - \delta_{k''', k'} - \delta_{k', k'''}k)}{N\langle f \rangle} (1 - P_{k''',(k',k)}) + \]
\[ \sum_{k', k'''} P(k', k + 1) \frac{f(k''')(N(k''') - \delta_{k''', k'} - \delta_{k', k'''}k+1)}{N\langle f \rangle} (1 - P_{k''',(k',k+1)}) \]

\[ P(k, t + 1) = P(k) + F(P(k, t), P(k, k', t), P(k, k', k'', t)) \]
Simple Graph Edge Selection Process

Master equation

\[ N(k, t + 1) = N(k) - \sum_{k', k''} P(k', k'') \frac{f(k)(N(k) - \delta_{k,k'} - \delta_{k,k''})}{N\langle f \rangle} (1 - P_{k,(k',k'')}) + \]

\[ \sum_{k', k''} P(k', k'') \frac{f(k-1)(N(k-1) - \delta_{k-1,k'} - \delta_{k-1,k''})}{N\langle f \rangle} (1 - P_{k-1,(k',k'')}) - \]

\[ \sum_{k', k'''} P(k', k) \frac{f(k''')(N(k''') - \delta_{k''',k'} - \delta_{k''',k})}{N\langle f \rangle} (1 - P_{k''',(k',k)}) + \]

\[ \sum_{k', k'''} P(k', k + 1) \frac{f(k''')(N(k''') - \delta_{k''',k'} - \delta_{k''',k+1})}{N\langle f \rangle} (1 - P_{k''',(k',k+1)}) \]

\[ P(k, t + 1) = P(k) + F(P(k, t), P(k, k', t), P(k, k', k'', t)) \]
Hierarchy of object distributions
Higher level objects

Conclusion

- Edge-colored class of models can be interesting option for some applications.
- The particular processes would need more research to improve and understand their behavior.
- The analytical solution can be very hard if the simple graph constraints are implicitly included.
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